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AN ELEMENTARY

TEXT-BOOK OF PHYSICS:

PART I.

GENERAL PHYSICS.

BY

R. WALLACE STEWART, D.Sc.(Lond.)

With 187 Illustrations.

LONDON:
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PREFACE.

This volume, although issued subsequently to Part II. "Sound," Part III. "Light," Part IV. "Heat," is presented as the opening one of the series forming "An Elementary Text-Book on Physics." It is written in the same exact, simple, and straightforward manner which has commended the other volumes and made them popular with Students who are preparing for any of the usual Elementary Examinations on Physics.

The presentation of the subject in separate volumes suited to the requirements of the Student was considered desirable, as it enabled the author to deal adequately with the fundamental facts and principles without the loss of interest always manifest when the whole subject is compressed into one small volume.

Teachers and reviewers have been unanimous in their praise of the earlier volumes, both in regard to the manner in which the subjects have been treated, and the excellent print and diagrams, which are new, and not mere rearrangements of the old stereotyped forms.

A melancholy interest attaches to this present volume, from the fact that the distinguished author died suddenly soon after the completion of the work.

October, 1910.

CONTENTS.

GENERAL PHYSICS.

CHAPTER I.

INTRODUCTORY.

1. **The Scope of Physics.**—The science of Physics may be said to deal with the phenomena of matter and ether.

Matter is the material or stuff of which are made all bodies which occupy space, and which are perceptible by us through the senses. It possesses certain fundamental properties, such as *inertia* and *gravitation*, which are dealt with later.

The **ether** is the medium which is assumed to fill all space, and to permeate all matter. It cannot be perceived by the senses, but there can be little doubt of its objective existence. Our knowledge of its existence and properties is entirely of an indirect character, derived from the study of the phenomena which are assumed to be associated with it, but this knowledge rests upon a very firm foundation of experimental evidence.

The relation between ether and matter is of too uncertain and speculative a character to be considered here. It may, however, be stated that it is in every way probable that matter is in some sense a modified form of ether, and that some of its properties are determined by its relation with the ether with which it is associated.

The branches into which the science of Physics is usually divided are Motion, Properties of Matter, Sound, Heat (including Radiation), Light, and Magnetism and Electricity.

The measurement of **space**, the measurement of **time**, and the measurement of **quantity of matter** are the fundamental measurements of the science. The distribution of matter in space, the specification of the relative position or *configuration* of any system of material bodies, and the consideration of the changes of configuration which take place in any system with time, are included under the general term **motion**. The study of **motion**, in this general sense, and the study of the general **properties of matter**, form the usual introduction to Physics.

Sound or **Acoustics** deals with the vibratory motion of bodies and with wave motion in *material media*, with special reference to the sense of hearing.

The phenomena of **Heat** are phenomena pertaining to *matter*, and include many important changes in the state and properties of matter. **Radiation** in a limited sense is the transverse wave motion set up in the *ether* by the vibratory motion of the molecules of a body, and **Light** is radiation within certain limits of wave length.

The phenomena of **Magnetism and Electricity** are essentially *ether phenomena* produced under conditions associated with the existence of certain states of strain or motion in the *ether*, and with the effects attending the presence of matter in the ether under these conditions.

The study of matter, in so far as it relates to the different kinds of elementary substances or *elements* which constitute matter, to the interaction of these elements with each other, and to the composition and properties of the compounds they form with each other, constitutes the science of **Chemistry**, and does not come directly within the scope of Physics.

2. **Physical Quantities.**—Physics is one of the exact sciences. Measurement is the basis of its experimental work and mathematical reasoning is the basis of its theory.

The physical quantities which are defined and measured experimentally will be explained as they arise in the exposition of the subject. It will be understood, however, that at the very outset quantities such as *length*, duration of *time*, and *mass* must be measured. These lead to other quantities, such as *area*, *volume*, *velocity*, and *acceleration*; and, as the subject develops, more complex quantities such as *force*, *work*, *energy*, and *power* are introduced. In the same way in every branch of the subject all measurable quantities are defined and measured so that the theoretical principles of the subject can be established on a sure foundation of exact quantitative knowledge.

3. **Units.**—The first essential in measuring any quantity is a suitable **unit** of measurement. A unit must evidently be of the same kind or denomination as the quantity to be measured, and its magnitude must be definitely specified either by direct reference to a *standard* in which it is realised, or by defining its relation to the standard or standards on which its value ultimately depends.

A quantity is measured in terms of a given unit by determining the number of times it contains the unit, and the magnitude of the quantity can then be expressed as n times the magnitude of the unit, or as equal to n **units**, where n is a number which may be a whole number or a fraction. For example, any *length* may be measured in terms of the *yard* as unit, and if a given length is found to be 5·3 times the length of a yard it is said to be 5·3 yards in length.

It will be seen that the magnitude of any quantity must always be given as n *units*, and in order that it may be completely specified, the value of n and the name of the unit must be definitely stated. The number n which gives the number of units in any magnitude is called the *measure* or *numeric* of the magnitude.

When the magnitude of the same quantity is given in terms of different units of the same kind the measures of the magnitude must evidently be different and must vary inversely as the relative magnitudes of the units employed. A length of *2 yards* may, for example, be expressed as *6 feet*, or *72 inches*. The measures here are in the ratio 1 : 3 : 36 when the relative magnitudes of the units are in the ratio $1 : \frac{1}{3} : \frac{1}{36}$. That is, the measure is *multiplied* by *n* when the magnitude of the unit is *divided* by *n*.

4. **Fundamental and Derived Units.**—The magnitude of the unit selected for the measurement of any quantity may be decided on purely arbitrary grounds, or it may be determined by a formal definition of the unit. If an arbitrary unit were selected for the measurement of every physical quantity, without any consideration of the inter-relations of the quantities, it would be found that endless confusion and trouble would result from the complicated relations between the units selected. For example, if any arbitrary length were selected as unit of length, and any arbitrary area as unit of area, it would be necessary, in finding the area of any regular plane figure by the rules of mensuration, to know the area of the square on the unit of length in terms of the arbitrary unit of area.

For this reason the plan has been adopted of selecting the units of certain quantities as **fundamental units**, and then deriving the units for all other quantities from these fundamental units by means of carefully framed definitions. The units derived in this way from the fundamental units are known as **derived units.**

It is found that in order to build up a system of units on this plan, the fundamental quantities need not be more than three in number, and may, in theory, be any three quantities. The units of these quantities, the fundamental units, may be selected as arbitrary units or determined by definition, but, in either case, *they must be capable of exact realisation as permanent standards of reference.*

The quantities adopted as fundamental physical quantities are **length, mass,** and **time,** so that the units of length, mass, and time are the fundamental units of the whole system of physical units. All the other units of the system are derived units. The unit of area, for example, is defined as the area of the square on the unit of length, and is, therefore, derived from the unit of length. Similarly the unit of volume is the volume of a cube having its edge of unit length, and is also derived from the unit of length. The unit of velocity is the unit of length per unit of time, and is, therefore, derived from the units of length and time. The unit of force is that force which, acting on unit mass, produces unit change of velocity in unit time, and is thus derived from the units of length, mass, and time. In the same way any other physical unit may be derived from one or more of the three fundamental units.

The quantities length, mass, and time are specially suitable for use as fundamental quantities. The units of length and mass are capable of very exact realisation as permanent and invariable standards, and the unit of time can be definitely specified in terms of the time in which the earth makes one complete revolution. The units of length and mass in general use are arbitrary units. An attempt was made by French physicists, as explained in Chapters iii. and v., to connect both these units with natural constants, and so to put them on the same basis as the unit of time. It was found, however, that the constants selected were not ascertained with sufficient accuracy* to enable

* This point may be made clearer by an example. Suppose the unit of length to be *specified* as the millionth part of the polar diameter of the Earth. In order to *realise* this unit as a permanent standard of reference it is necessary to construct a bar of platinum or some permanent metal of the specified length. This means that *the length of the polar diameter must be accurately known*, and that the bar must be constructed so as to be exactly equal to the millionth part of this known length. If the length of the bar is derived from an inaccurate value of the polar diameter, it must evidently be reconstructed when a more accurate value is found, or, if the old standard is retained, the specification of the unit must be changed.

the reference standards to be constructed in exact accordance with the specifications of the units.

A system of units built up, in this way, of units derived from certain fundamental units, is called an *absolute system of units* or a system of **absolute units.**

The system of absolute units in general use in physics is the **C. G. S. system,** in which the fundamental units of length, mass, and time are the Centimetre, the Gramme, and the Second respectively.

The "English" system of absolute units, sometimes called the F. P. S. system, is a system in which the Foot, the Pound, and the Second are the fundamental units of length, mass, and time respectively. It is still used in text books on theoretical mechanics, but is seldom used in physics.

5. **The Metric System.**—The **metric system** is not a system of units in the sense explained in the foregoing article. It is practically a set of "measures" in which the **decimal system** is employed in forming the multiples and sub-multiples of the units selected.

The notation of the system is the same in every table. The multiples of the unit by 10, 100, and 1,000 are designated by placing the Greek prefixes **deca-**, **hecto-**, and **kilo-** before the unit; and the corresponding sub-multiples are designated by placing the Latin prefixes **deci-**, **centi-**, and **milli-** before the unit. The multiple by 10,000, which is sometimes used, is distinguished by placing the prefix **myria-** before the unit. Thus we have the following scheme in each table :—

	Myria- (unit)	=	10,000 units.
Multiples,	Kilo- (unit)	=	1,000 units.
	Hecto- (unit)	=	100 units.
	Deca- (unit)	=	10 units.
	(Unit)		1 unit.
Sub-multiples,	Deci- (unit)	=	·1 unit.
	Centi- (unit)	=	·01 unit.
	Milli- (unit)	=	·001 unit.

The units of the system are given below :—

QUANTITY MEASURED.	NAME OF UNIT.
Length, . . .	**Metre.**
Area, ...	The square metre.
Area (Land Measure), .	The **are** or square decametre.
Volume, . . .	The cubic metre.
Capacity, . . .	**Litre** or cubic decimetre.
Mass, ...	**Gramme.**

The system takes its name, the *metric system*, from the *metre* the unit of length.

CHAPTER II.

SCALAR AND VECTOR QUANTITIES.

6. **Scalar Quantities.**—A quantity which possesses magnitude only, and is, therefore, completely specified by its magnitude, is known as a **scalar** or **scalar quantity**. Thus, area, volume, time, mass, and other quantities to be dealt with later, are scalar quantities.

Scalar quantities of the same kind are added and subtracted by the ordinary arithmetical rules, or they may be assigned positive and negative signs according to some recognised convention, and treated as algebraic quantities.

7. **Vector Quantities.**—A quantity which possesses direction as well as magnitude, is known as a **vector** or a **vector quantity**.

The distance of one point from another is a vector quantity, and it will be found later that other quantities, such as displacement, velocity, acceleration, and force, are vector quantities.

In dealing with vector quantities it is necessary to be able to add and subtract them in such a way as to take account of the direction as well as the magnitude of the quantities. Ordinary arithmetical addition and subtraction take account of magnitude only. Algebraic methods make provision for the addition and subtraction of quantities which may have one of two opposite directions denoted respectively by a positive and negative sign. A special method must therefore be found for the addition and subtraction of vector quantities.

8. **Composition of Vector Quantities.**—The usual method of adding and subtracting vector quantities is a graphical one.

A straight line may represent any vector quantity if the number which measures the length of the line is the same as the number which measures the magnitude of the quantity, and if the direction of the line represents the direction of the quantity. The vector quantities to be added or subtracted may thus be represented by straight lines, and addition or subtraction becomes a graphical process.

The process of adding or *compounding* vector quantities is known as the *composition* of the quantities, and the sum of the added quantities is called the *resultant*. It is obvious that only quantities of the same kind can be compounded, and that the resultant is a quantity of the same kind as the quantities compounded.

The rule for compounding two vector quantities by this method may be given in the following terms.

From any point A, Fig. 1, draw two straight lines AB and AC, to represent the quantities in magnitude and direction. Then complete the parallelogram, ABDC, of which these two lines are adjacent sides, and draw the diagonal AD from the point A. This diagonal, AD, now represents the resultant of the two quantities in the same way as the lines AB and AC represent the quantities themselves. That is, the number which measures the length of AD is the number which measures the magnitude of the resultant, and the direction of AD represents the direction of the resultant.

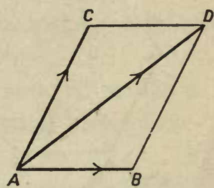

Fig. 1.

This form of the rule may be called the *parallelogram rule*. Another form known as the *triangle* rule may be derived from it. It will be seen in Fig. 1 that the line BD is the same in magnitude and direction as AC. The lines AB and BD, therefore, represent the quantities to be compounded, and AD represents their resultant. Hence, if starting from any point A, we draw, one after the other, in order, two lines, AB and BD, to represent the quantities to be compounded; then AD, the third side of the triangle ABD, drawn from the starting point A to the finishing point D, represents the resultant of the two quantities.

This is the triangle form of the rule for the composition of two vector quantities. It has the advantage that it can be extended in the same terms to provide a rule for the composition of any number of vector quantities. Thus, if we wish to compound five vector quantities of the same kind we draw from any starting point A (Fig. 2) five lines, AB, BC, CD, DE, and EF, in successive order to represent the five quanti-

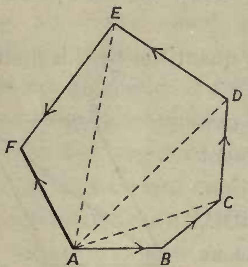

Fig 2.

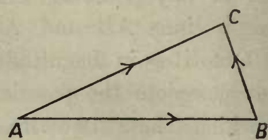

Fig. 3.

ties in magnitude and direction. The line AF, drawn from the starting point A to the finishing point F of this sequence of lines, then represents the resultant of the five vector quantities in the same way as the lines AB, BC, CD, DE, and EF represent the quantities themselves. This rule, known as the *polygon* rule, is evidently an extension of the triangle rule, for, by the triangle rule, the resultant of the quantities represented by AB and BC is represented by AC. Similarly, AD represents the resultant of the quantities represented by AC and CD, or the resultant of the three quantities represented by AB, BC, and CD. In the same way it follows that AE represents the resultant of the quantities represented by AB, BC, CD, and

DE, and that AF represents the resultant of the quantities represented by AB, BC, CD, DE, and EF.

The method of finding the difference of two vector quantities is most conveniently derived from the triangle rule for compounding two quantities.

Thus, in Fig. 1, since the resultant of the two quantities represented by AB and BD is represented by AD, it follows that the difference between the two quantities represented by AD and AB is represented by BD.

The line BD (with the arrow from B to D) represents the difference obtained by subtracting the quantity represented by AB from the quantity represented by AD, while the line DB (with the arrow from D to B) represents the difference obtained by subtracting the quantity represented by AD from the quantity represented by AB.

Hence, we have the following rule for finding the difference of any two vector quantities of the same kind.

From any point A, Fig. 3, draw lines AB and AC to represent in magnitude and direction the two quantities whose difference is required; then join BC.

The line BC, drawn from B to C, then represents the difference obtained by taking the quantity represented by AB from the quantity represented by AC.

9. **Resolution of a Vector Quantity.**—Just as a number (such as 12) may be split up arithmetically into an infinite number of pairs of numbers (such as 9 and 3, or 8 and 4, or 10 and 2, or 5·9 and 6·1, &c.), which, when added together, make up the number as their sum, so any vector quantity may be **resolved** into an infinite number of pairs of **components** which, when compounded together, make up the given quantity as their resultant. This may be done by direct application of the parallelogram or triangle rule. Let AB, Fig. 4, represent any vector quantity; then, if we construct *any* parallelogram, ACDB, on AB *as diagonal*, the lines AC and AD represent two

quantities which, if compounded together, make up the quantity represented by AB as their resultant. That is, the quantity represented by AB is resolved, or split up into two components represented by AC and AD. Since this is true for *any* parallelogram constructed on AB as diagonal, it is evident that the quantity represented by AB may, in this way, be resolved into any number of pairs of components.

In the same way with the triangle rule, if AB, Fig. 5, represent any vector quantity, and ACB be *any* triangle constructed on AB as base, the lines AC and CB represent quantities which, if compounded together, make up the quantity represented by AB as resultant. That is, the quantity represented by AB is resolved into two components represented by AC and CB.

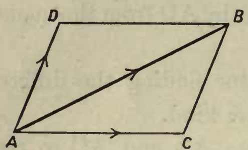

Fig 4.

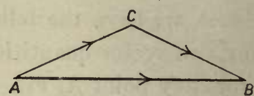

Fig. 5.

If it is required to resolve any given quantity into two components, of which one is given, it is evidently sufficient to find the difference between the two given quantities, as explained above. Thus, if AB, Fig. 5, represent the given quantity, and AC the known component, then CB represents the other component, for the quantities represented by AC and CB have the quantity represented by AB as their resultant.

If it is required to resolve a given quantity into two components in given directions, it will be found most convenient to apply the parallelogram rule. Thus, let AB, Fig. 6, represent the given quantity, and AX and AY the given directions for the required components. Through B draw the lines BC and BD, parallel respectively to AY and AX, and cutting these lines at the points C and D. Then the lines AC and AD obviously represent the required components.

An important case of the *resolution* of a vector quantity into components, is that in which the given quantity is to be resolved into two components at right angles, one of which is required to be in a given direction, and the other at right angles to it. For example, let it be required to resolve the given quantity, represented by AB in Fig. 7, into two *rectangular components*, one of which is to be in the direction AX, and the other at right angles to AX.

From B draw BC perpendicular to AX, and cutting it at C, and complete the rectangle ACBD. The lines AC and AD now represent the required components; the component represented

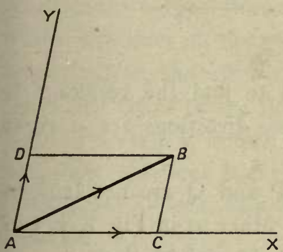

Fig. 6.

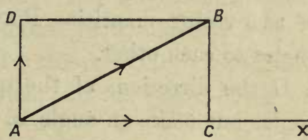

Fig. 7.

by AC has the given direction AX, and the other, represented by AD, has a direction at right angles to AX.

If the angle BAX, the angle between the direction of the given quantity represented by AB and the given direction AX, be denoted by a it will be seen that

$$\frac{AC}{AB} = \cos a, \text{ or } AC = AB \,.\, \cos a,$$

and

$$\frac{AD}{AB} = \frac{CB}{AB} = \sin a, \text{ or } AD = AB \,.\, \sin a.$$

That is, if R denote the magnitude of the given quantity represented by AB, and P and Q the magnitudes of the components represented by AC and AD respectively, we have

$$P = R \,.\, \cos a$$

and $$Q = R \,.\, \sin a,$$

where a is the angle between the direction of the given quantity R and the given direction of the component P.

It is also evident, by application of *Euc.* i. 47 to the figure, that

$$AB^2 = AC^2 + CB^2 = AC^2 + AD^2,$$

or $$R^2 = P^2 + Q^2.$$

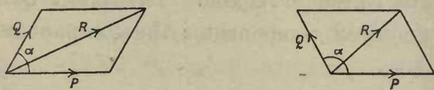

Fig. 8.

This relation may evidently be applied to find the resultant R of two vector quantities, P and Q, whose directions are at right angles to each other.

If the directions of the quantities P and Q are not at right angles but make an angle, a, with each other as in Fig. 8, it can easily be proved with the aid of *Euc.* ii. 12 and 13 that

$$R^2 = P^2 + Q^2 + 2PQ \cos a.$$

CHAPTER III.

MEASUREMENT OF LENGTH, AREA, AND VOLUME.

10. **Units of Length.**—The legal units of length in England are the **yard** and the **metre**. The **yard** is defined by Act of Parliament as the distance at 62° F. between the centres of the transverse lines on the two gold plugs in a bronze bar, originally deposited in the office of the Exchequer, but now kept at the Standard Office of the Board of Trade. The length thus defined and preserved is known as **the standard yard**, and copies of the standard are kept at the Houses of Parliament, the Royal Mint, the Royal Observatory at Greenwich, and the Royal Society of London.

The multiples and sub-multiples of the yard are given in the familiar table of English long measure. We need only notice here the **foot** and the **inch** as sub-multiples and the **mile** as a multiple.

Thus we have:

12 inches = 1 foot.
3 feet = 1 yard.
1760 yards = 1 mile.

It will be understood that, although the yard is the **legal** unit or standard of length, any one of its sub-multiples or multiples, or, in fact, any specified part of it, may be taken as the unit of length in any set of length measurements.

The **metre** is really the legal unit of length in France, but it was legalised in England in 1897. It is defined as the distance at 0° C. between the **ends** of a platinum rod constructed by Borda.

The metre was intended originally to be the ten-millionth part of the distance from the North Pole to the Equator measured along the meridian passing through Paris. Very careful measurements were made for the purpose of determining this distance, and Borda constructed the platinum bar now used as the standard metre to be the ten-millionth part of the result then obtained. Later measurements, however, show that the distance from the North Pole to the Equator along the meridian of Paris is rather more than ten million times the length of Borda's bar, so that the metre is no longer defined as the ten-millionth part of this quadrant on the earth's surface, but simply as the distance at 0° C. between the ends of Borda's platinum bar.

Although the metre may be legally used in England as a standard of length it is not yet in common use in this country. It is, however, the standard most generally adopted in other European countries.

The multiples and sub-multiples of the metre are given below

Kilometre (km.)	=	1000	metres.
Hectometre	=	100	,,
Decametre	=	10	,,
Metre			
Decimetre (dm.)	=	·1	metre.
Centimetre (cm.)	=	·01	,,
Millimetre (mm.)	=	·001	,,

Of the multiples only the kilometre is in general use; it is used on the Continent for specifying distance from place to place in the same way as we use the mile in England.

The sub-multiples are all in use, but the centimetre and millimetre are most generally used.

The unit of length generally adopted for scientific measurements is the **centimetre.** The foot and the inch are occasionally used in England for certain measurements, but it is now practically the universal custom to employ the centimetre as the unit of length in all physical measurements.

The relative magnitude of the yard and the metre is given with sufficient accuracy by the following equivalents:—

1 metre = 39·37 inches = 1·0936 yards.
1 yard = 0·9144 metre = 91·44 cms.

These equivalents may be reduced to the following approximate values which are convenient for general use.

1 metre = 3·28 feet.
1 foot = 30·48 cms.
1 inch = 2·54 cms.

For rough calculations it is convenient to remember that

10 cms. = 4 inches (nearly),
or, more exactly, 33 cms. = 13 inches;
also that a millimetre is slightly less than the twenty-fifth of an inch.

The relation between the mile and the kilometre is given by the equivalent:

1 kilometre = ·62137 mile.
1 mile = 1·60935 kilometres.

That is, a kilometre is nearly five-eighths of a mile, so that five miles is, roughly, equal to eight kilometres.

11. **Measuring Scales.**—The measuring scale in common use for physical measurements is the **metre scale** one metre in length. It is usually made of box-wood or metal, and is generally graduated to show decimetres, centimetres, and millimetres along one measuring edge, and for convenience inches and tenths of an inch along another edge. Small steel scales showing a variety of small divisions of the inch and centimetre are also in use.

The most accurate metre scales are made in metal. Steel, brass, and gun-metal have been used for this purpose, but it is probable that the new metal "invar," an alloy of nickel and steel, will be generally used in future. This metal has a very small coefficient of expansion with change of temperature, so that a scale

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made of it would not be subject to any appreciable error due to change of length with change of temperature.

Metre scales are generally made as *line scales*—that is, the graduated metre extends from a *line* near one end to a *line* near the other end. Some scales are, however, made as *end scales*, the graduated metre extending from one end of the scale to the other. Borda's standard metre, for example, is an *end* measure, while the standard yard is a *line* measure.

12. **The Vernier.**—In certain cases a length may be measured with sufficient accuracy by simply applying the measuring edge of a suitable scale directly to it, and then reading off the required length from the graduations of the scale. With a scale graduated in tenths of an inch it is possible in this way, by *estimating tenths* of a scale division, to measure a length with fair accuracy to one-hundredth of an inch. Similarly, with a scale graduated in millimetres it is possible with care and practice to read to a tenth of a millimetre or one-hundredth of a centimetre.

It is not possible to attain to greater accuracy than this by any further subdivision of the scale, so that when greater accuracy is required other methods have to be adopted. Of these methods, the vernier method is the simplest and most commonly used.

A **vernier** is a short auxiliary scale used with the measuring scale for the purpose of reading the scale to some particular fraction of a scale division. The general principle on which a vernier is constructed may be stated concisely in the following way. If it is desired to construct a vernier to read to $\frac{1}{n}$ of a scale division, a length equal to $(n - 1)$ or $(n + 1)$ scale divisions is taken and divided into n equal parts to give the vernier divisions. Thus, if we wish to make a vernier to read to $\frac{1}{20}$ of a scale division we mark off a length equal to 19 (or 21) scale divisions and divide it into twenty equal parts. The small scale thus obtained would be a vernier scale which would enable readings to be taken on the measuring scale to one-twentieth of a scale division.

It will be seen that by this method of construction a division on a vernier reading to $\frac{1}{n}$ of a scale division differs from a scale division by $\frac{1}{n}$ of a scale division; it is either the $\frac{1}{n}$ part of a scale division less, or the $\frac{1}{n}$ part greater than a scale division, according as it is made on the $(n - 1)$ or $(n + 1)$ plan explained above. It is generally most convenient to make a vernier on $(n - 1)$ plan so that its divisions are less than the scale divisions by $\frac{1}{n}$ of a scale division. This difference between the length of a vernier division and the length of a scale division given as a fraction of a scale division is known as the *least count* of a vernier.

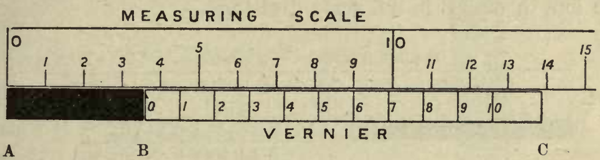


Fig. 9.

The method of using a vernier can now be explained with the help of the following example.

Suppose it is required to measure the length of the rod AB by means of the measuring scale and vernier shown in Fig. 9. It will be seen from the figure that the length of the vernier is equal to 9 scale divisions, and that it is divided into 10 equal parts; it therefore reads to $\frac{1}{10}$ of a scale division, or its least count is $\frac{1}{10}$ of a scale division.

When the measuring scale is applied directly to AB, as shown in the figure, the length of AB is found to be greater than three and less than four scale divisions. The vernier is then applied, as shown at BC, so that *the zero division is placed exactly at the point whose position on the scale is to be determined.* The length of AB is thus seen to be equal to three scale divisions and the portion of the fourth division which lies to the left of the zero of the vernier.

If we now pass along the vernier from the zero to the sixth division we see that the distance between each successive division and the scale division immediately to the left of it gets less and less, until, at the sixth division of the vernier, the two divisions are coincident. This evidently indicates that the distance from the zero of the vernier to the first scale division to the left of it (3) is 6 tenths of a scale division. For, from the construction of the vernier the corresponding distance at the fifth vernier division is 1 tenth of a scale division, at the fourth division it is 2 tenths, at the third 3 tenths, at the second 4 tenths, at the first 5 tenths, and at the zero 6 tenths. That is, the length of AB is 3·6 scale divisions.

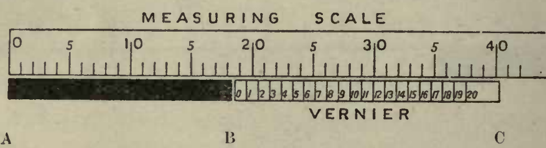


Fig. 10.

In the same way it can be made out that the length of the rod AB in Fig. 10 is $18\frac{11}{20}$ scale divisions, or 18·55 scale divisions.

The manner in which the vernier divisions are marked and numbered should be noted. It will be seen that the numbering is in the same direction on the vernier as on the scale.

The principle of the vernier is applied in a number of measuring instruments. The most important of these is the **vernier callipers** shown in Fig. 11. The purpose of the instrument is sufficiently indicated for general purposes in the figure. The details of its construction and the method of using it can be learnt satisfactorily only by practice in the laboratory.

13. **The Micrometer Screw.**—Another important method of attaining great accuracy in the measurement of length depends upon the use of the micrometer screw. This is merely a very accurately cut screw of small pitch, provided with a large head, which is divided round its circumference into a convenient number of equal parts arranged so that any fraction of a complete turn of the screw can be measured with an accuracy which depends upon the size of the head and the number of divisions into which its circumference is divided. It is called a *micrometer* screw because it is capable of measuring very small differences in length.

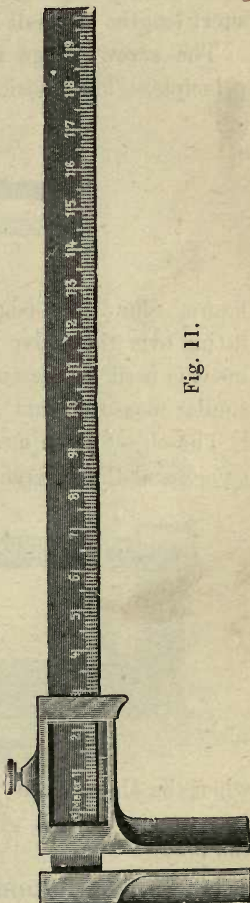


Fig. 11.

The theory of the micrometer screw is simple. It is evident that for one complete turn of the screw the point moves through a distance equal to the pitch of the screw. Hence, if by dividing the head of the screw into 10, 100, or 1,000 parts, we can measure the tenth, hundredth, or thousandth part of a complete turn, we can measure by means of the point of the screw to one-tenth, one-hundredth, or one-thousandth of the pitch of the screw. For example, if the pitch of the screw is $\frac{1}{2}$ mm., and the head is divided into 500 parts, the turning of the screw through one of these parts causes a displacement of the point of the screw through $\frac{1}{500}$ of $\frac{1}{2}$ mm., or ·001 mm. A screw constructed in this way would, therefore, be able to indicate a difference in the position of its point equal to the thousandth part of a millimetre.

Fig. 12 indicates diagrammatically the manner in which the principle of the micrometer screw can be adapted for measuring short lengths or small differences in length.

The **screw gauge** shown in Fig. 13 is constructed on this principle. The divisions of the head are marked, as seen in the

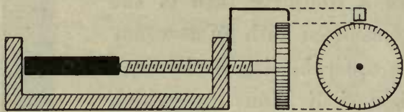


Fig. 12.

figure, round the edge of a sleeve carried by the head, and fitting over the collar in which the screw works. The instrument is used for measuring the diameters of wires and for other similar measurements.

The object to be measured is placed between the jaws of the gauge, and the difference between the reading of the screw head

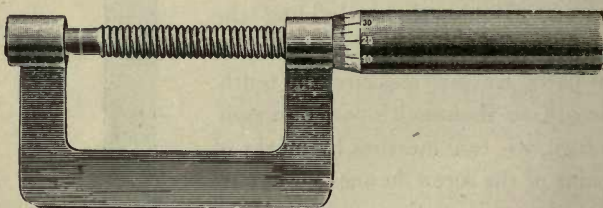


Fig. 13.

when the object is in position, and the zero reading when the jaws are in contact, gives the thickness of the object between the jaws.

14. **The Cathetometer.**—The cathetometer is an instrument used for measuring differences in vertical height.

A simple form of the instrument is shown in Fig. 14. It consists essentially of (*a*) a vertical scale engraved on, or attached to a vertical metal rod, and (*b*) a telescope carried by a slide, which can be moved up and down the rod. This slide can be clamped at any point on the rod, and its position on the scale can then be read accurately by means of a vernier which moves with it over the scale.

The telescope is generally mounted on the slide in such a way that it can be rotated round the rod as axis, and also round a horizontal axis, so that its line of sight can be elevated or depressed as required. It carries a spirit level in order that it may, when required, be set in a horizontal position.

The metal rod or pillar which carries the telescope is usually mounted on a heavy base provided with levelling screws, in order that it may be possible to adjust the rod and scale in a vertical position.

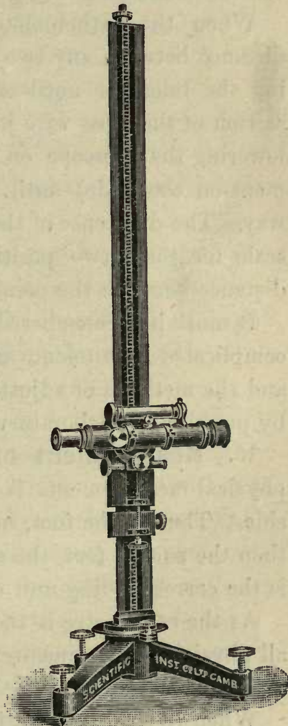


Fig. 14.

In order to obtain accurate readings with the instrument, it is essential (1) that the scale rod should be vertical; and (2) that in any given measurement, or set of measurements, the axis of the telescope should make the same angle with the horizontal at all points on the scale at which readings are taken. The first adjustment is readily made by adjusting the levelling screws of the base until the telescope, after being set horizontal in any position, remains horizontal throughout a complete revolution round the rod as axis. The second condition is ensured if the telescope is horizontal in all positions, but it is evidently fulfilled (when the scale rod is vertical) for any position of the telescope relative to the horizontal, so long as the adjustments of the telescope remain unchanged throughout the readings.

When the cathetometer is properly adjusted the vertical distance between any two points is readily measured by adjusting the telescope until one of the points is seen at the intersection of the cross wire in the field of view, and then raising or lowering the telescope on the scale (without altering its adjustment on the slide) until the second point is seen in the same way. The difference of the readings of the vernier on the vertical scale for these two positions, then gives the required vertical distance between the points.

It must be remembered that a good cathetometer is a very complicated instrument, and the full details of its construction and the methods of adjusting it and using it can only be learnt by practice in a laboratory.

15. **Measurement of Area.**—The unit of area in all physical measurements is the square on the unit of length as side. Thus, if the foot, inch, or centimetre is the unit of length, then the **square foot**, the **square inch**, or the **square centimetre** is the corresponding unit of area.

As the centimetre is the unit of length generally adopted in all physical measurements, the **square centimetre** is the unit of area in most general use.

When the unit of area is derived in this way from the unit of length, it is evident that the multiples and sub-multiples of the unit of area can be derived from the corresponding multiples and sub-multiples of the unit of length. Thus, since the square foot is the square on a side 1 foot or 12 inches long, it must contain 12^2 or 144 square inches. Similarly, since 1 centimetre = 10 millimetres, we must have

$$1 \text{ sq. cm.} = 10^2 \text{ sq. mms.} = 100 \text{ sq. mms.};$$

and, in the same way,

1 square decimetre = 10^2 sq. cms. = 100 sq. cms.

and, 1 square metre = 100^2 sq. cms. = 10,000 sq. cms.

The numerical relations between the square foot or the square inch, and the square cm. can be derived from the linear relations already given. The following equivalents may, however, be given :—

1 sq. foot = 929·04 sq. cms.
1 sq. in. = 6·4517 ,,
1 sq. cm. = ·155 sq. inch.

It is convenient to remember that 31 square inches is almost exactly equal to 200 square centimetres.

The measurement of the area of any regular figure resolves itself into measurement of length. The necessary dimensions of the figure are measured by some suitable method of length measurement, and the area is calculated by the appropriate rule in mensuration.

The area of any irregular figure can be found approximately by transferring the figure to squared paper, and counting the number of small squares of known area which are enclosed by it. It can also be found with fair accuracy by cutting out the figure in thin foil or cardboard of uniform thickness, and then comparing the weight of this piece of foil or cardboard with the weight of a known area of the same material.

The area of an irregular plane figure can be measured accurately by means of a mathematical instrument known as the *Planimeter*. The theory and construction of this instrument are, however, beyond the scope of this work, and cannot here be considered.

16. **Measurement of Volume.** — The unit of volume adopted in physical measurement is the cube on the unit of length as edge. Thus if the foot, inch, or centimetre is taken as the unit of length, the **cubic foot,** the **cubic inch,** or the **cubic centimetre** is the corresponding unit of volume.

The centimetre being the general unit of length the **cubic centimetre** is the unit of volume in general use in all physical measurements.

When the unit of volume is derived in this way from the unit of length the multiples and sub-multiples of the unit of volume can be derived, as in the case of the units of area, from the corresponding multiples and sub-multiples of the unit of length. Thus, since the cubic foot is a cube of 1 foot or 12 inches edge, it must contain 12^3 or 1,728 cubic inches. Similarly, since a cubic centimetre is a cube of 1 centimetre or 10 millimetres edge it must contain 10^3 or 1,000 cubic millimetres. In the same way we have

$$1 \text{ cubic decimetre} = 10^3 \text{ cub. cms.} = 1{,}000 \text{ cub. cms.},$$

or,

$$1 \text{ cubic metre} = 100^3 \text{ cub. cms.} = 1{,}000{,}000 \text{ cub. cms.}$$

The relative magnitude of the cubic inch or the cubic foot and the cubic centimetre is conveniently expressed by the following equivalents—

1 cub. inch = 16·388 cub. cms.
1 cub. cm. = ·06102 cub. in.

From these values it will be seen that 1,000 cub. cms. is only very slightly greater than 61 cubic inches.

The cubic centimetre is also the unit of *capacity* generally used in measuring the volume of a liquid or a gas in all scientific measurements.

The **litre**, the unit of capacity adopted in the metric system of units, is a cubic decimetre or 1,000 cubic centimetres. It is the unit in which large volumes of a liquid or a gas are generally expressed.

The English units of capacity are seldom used in scientific measurements. The **gallon** is the volume occupied by 10 pounds of pure water at 62° F., and is equal to 277·274 cubic inches.

The sub-multiples of the gallon, the pint, and the quart are defined by the relation

$$1 \text{ gallon} = 4 \text{ quarts} = 8 \text{ pints}.$$

The fluid ounce is the volume occupied by one ounce of pure water at 62° F., and is, therefore, the $\frac{1}{160}$ part of a gallon. It follows from this that a pint is equal to 20 fluid ounces.

The measurement of the volume of any regular solid resolves itself, as in the measurement of area, into measurement of length. The necessary dimensions of the solid are measured by some suitable method of length measurement, and the volume of the solid is then calculated by the appropriate rule of mensuration.

The volume of an irregularly shaped body can be measured by measuring the volume of water which it displaces, or, much more accurately, by determining its apparent loss of weight when weighed in water, or in some liquid in which it is insoluble, as explained later.

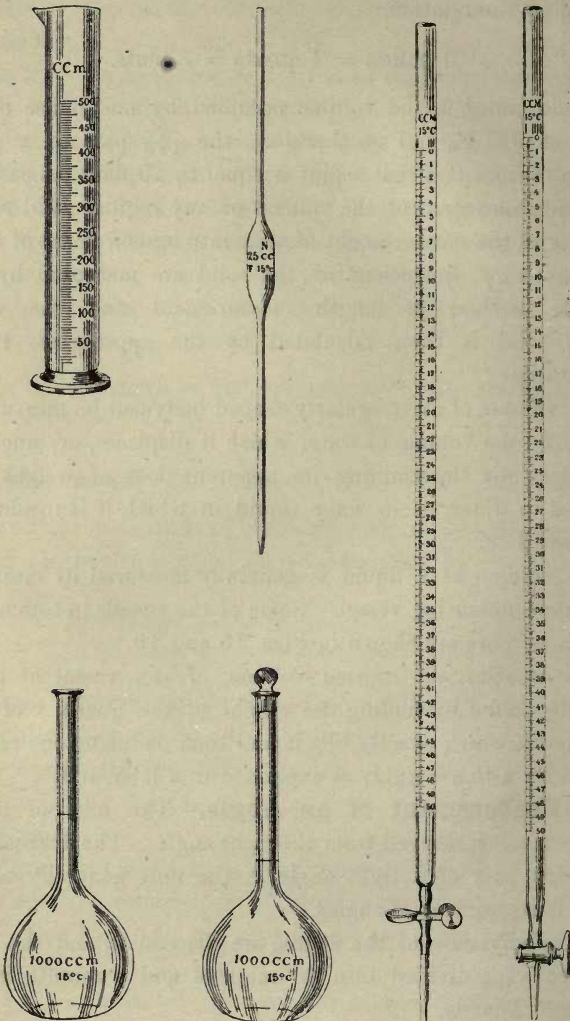
The volume of a liquid is generally measured by means of a graduated measuring vessel. Some of the vessels in common use for this purpose are shown in Figs. 15 and 16.

The capacity, or internal volume, of any vessel or tube is generally found by finding the weight of the quantity of water or mercury which exactly fills it and then deducing the required volume from this weight, as explained in a later article.

17. **Measurement of an Angle.**—The unit of angular measurement is derived from the right angle. The **degree,** which is the $\frac{1}{90}$ part of a right angle, is the unit generally adopted in the measurement of angles.

The subdivisions of the degree are the *minute*, and the *second*, a degree being divided into 60 minutes and a minute into 60 seconds. That is

60 seconds = 1 minute.
60 minutes = 1 degree.



Figs. 15 and 16.

or, in the usual notation,

$$60'' = 1'$$
$$60' = 1^\circ$$

The usual method of measuring an angle in practice is by means of a divided circular scale. The arc of a circle is proportional to the angle which it subtends at the centre of the circle, so that by subdividing the circumference of a circle into 360 equal parts we subdivide the four right angles at the centre of the circle into 360 degrees.

A circular scale for the accurate measurement of angles is generally divided to show divisions less than a degree, and is also provided with a vernier reading to some convenient fraction of a scale division. A common form of scale, for example, is divided into 20′ divisions, and carries a vernier reading to $\frac{1}{20}$ of a division. With this scale an angle at the centre of the scale could be measured to the nearest minute.

For theoretical purposes an angle is frequently measured by the ratio of the arc which it subtends at the centre of a circle to the radius of the circle. This ratio gives what is called the **circular measure** of the angle.

Since the ratio, $\frac{\text{arc}}{\text{radius}}$, which gives the circular measure of an angle is of unit value when the arc is equal to the radius, it follows that the unit of circular measure is the angle which is subtended at the centre of the circle by an arc equal in length to the radius. This unit is called a **radian.**

A right angle at the centre of a circle is subtended by an arc equal to one-fourth of the circumference. If r denote the radius of the circle, then $2\pi r$, where $\pi = 3{\cdot}1416$, is its circumference, and $\frac{\pi r}{2}$ is one-fourth of the circumference. The circular measure of a right angle is, therefore, given by the ratio $\frac{\pi r}{2} \Big/ r$ or $\frac{\pi}{2}$.

That is,

$$\frac{\pi}{2} \text{ radians} = 90^\circ,$$

or
$$1 \text{ radian} = \frac{90^\circ \times 2}{\pi} = 57{\cdot}2958^\circ,$$

or
$$1 \text{ radian} = 57^\circ\ 17'\ 44{\cdot}9''.$$

It should be noted that an angle is not a physical quantity. It is merely a number, the ratio of two lengths, and its measurement is, in practice, essentially a measurement of length.

CHAPTER IV.

MEASUREMENT OF TIME.

18. **Units of Time.**—The standard unit of time is derived from the period of rotation of the earth on its axis.

The interval of time that elapses between two successive transits of a fixed star across the meridian of any place is almost exactly equal to the period of time in which the earth makes one complete revolution.

The star is so distant that the direction in which it is seen from the earth is practically the same from all points on the earth's orbit round the sun, and the period between two successive transits of the star is, therefore, practically equal to the period of the earth's rotation on its axis.

This period of time is known as a *sidereal day* and is the astronomical unit of time.

The interval of time that elapses between two successive transits of the **sun** across the meridian of a place is not, however, equal to the time of one complete revolution of the earth on its axis, and is found also to vary from day to day. One reason for this is indicated in Fig. 17. Let A, B, and C represent the positions of the earth in its orbit round the sun, S, at noon on three consecutive days, at the place marked by the small arrow in the figure. If we consider the direction of the earth's rotation on its axis, as shown in the figure, we can see that the interval of time that elapses between the instant the earth is at A, with the small arrow pointing towards S, and the instant it is at B, with the small arrow again pointing to S, is the time taken by the earth in rotating through one complete revolution and the angle ASB. Similarly, the interval between the positions B and C is the time in which the earth rotates through a complete revolution and the angle BSC. Now the angles ASB and BSC are not equal, for the earth moves with variable velocity in an elliptical orbit round the sun, and the line joining it to the sun does not sweep out equal angles in consecutive days. It follows from this that the interval of time which elapses between two successive transits of the sun across the meridian of a place, or, in other words, the interval between noon by the sun on two successive days at any place is greater than the time in which the earth makes one complete revolution on its axis and varies from day to day throughout the year. Another cause of variation which produces a similar effect to a smaller degree is the inclination of the earth's axis to the plane of its orbit round the sun.

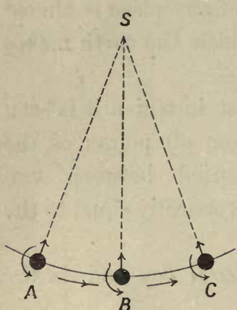


Fig. 17.

If, however, we take the average or mean value of this interval for a complete year, we get a definite interval of time known as the **mean solar day.**

The mean solar day is divided into 24 hours, the hour into 60 minutes, and the minute into 60 seconds.

A **mean solar second** is, therefore, the $\frac{1}{86,400}$ part of a mean solar day.

This unit, the mean solar second, is the unit of time generally adopted in all physical measurements.

The sidereal day is obviously less than the mean solar day, and is calculated to be equal to 23 hours 56 minutes 4·09 seconds of mean solar time.

19. **Instruments used for the Measurement of Time.**—The instrument generally used for the measurement of time is a clock or watch.

The mechanism and construction of a clock or watch cannot be considered here in any detail.* The following general points may, however, be noted. The mechanism of any instrument of this kind consists of four essential parts—(*a*) the *mainspring* or *weights*, from which the motive power is derived; (*b*) the *pendulum* or balance wheel, which determines by its motion the rate at which the mechanism moves; (*c*) the *escapement*, by which the pendulum or balance wheel is maintained in motion and is able, at the same time, to control and regulate the action of the motive power; and (*d*) the train of wheels by which the indicating hands are rotated.

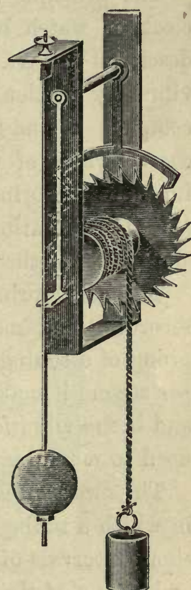


Fig. 18.

A simple form of escapement, known as *the dead beat escapement*, is shown in Fig. 18. In this figure the essential parts of the mechanism are easily distinguished. It is so constructed that the pendulum receives the successive impulses which maintain it in motion once during each swing, at the instant when it is at the middle point of its swing. These impulses are communicated by the teeth of the escapement wheel through the crutch and pendulum fork directly to the pendulum, and as they are communicated only when the pendulum is at the middle point of its swing, they have no disturbing effect on the time of swing. The teeth of the escapement wheel are thus allowed to "escape" from the pallets of the crutch at regular equal intervals, which are determined by the time of swing of the pendulum.

An accurately made clock controlled by a pendulum, and made to go with practically perfect regularity, is called a standard clock. It may be regulated to indicate astronomical time with

* See Ball's *Experimental Mechanics*, Lecture xx.

24 hours to the sidereal day, or mean solar time, in the usual way.

An accurately made clock or watch controlled by a balance wheel, and constructed to go with the most perfect regularity possible, is called a *chronometer*. It is the instrument generally used in navigation and is usually regulated to indicate mean solar time.

The principle involved in the regulation of the motion of a clock or watch by means of a pendulum or balance wheel is described in Art. 43. It is there explained that any body in vibratory motion executes each complete vibration, or each complete to-and-from movement, in the same time. This characteristic of vibratory motion, known as *isochronism*, is the fundamental principle of clock and watch construction. The period of vibration of the pendulum or balance wheel determines and regulates the rate at which the clock goes.

The same principle is applied in the use of a tuning-fork for recording and measuring very short intervals of time. If the prong of a tuning-fork in vibration makes n complete vibrations per second it makes every complete vibration in $\frac{1}{n}$ of a second, and if the vibrations are recorded in any suitable way they can be used to measure very short intervals of time.

The *chronograph*, shown in Fig 19, is a simple form of apparatus in which a tuning-fork is used in this way to record and measure short intervals of time. A light metal style is attached to the end of one of the prongs of the fork, and the fork is mounted so that the tip of the style rests lightly on the surface of a sheet of smoked paper rolled round the large drum shown in the figure. This drum is rotated by hand or by clockwork, and the point of the style traces a line on the smoked surface on which it rests as the drum rotates. If the fork is not in vibration this line is a straight line, but if it is in vibration the line has a characteristic wavy form, in which the length of each wave corresponds to the period of a complete vibration.

The line thus traced on the smoked surface may evidently be used for measuring the interval of time between any two instants if the points on the trace corresponding to these instants can be marked. This might be done by arranging for the drum to receive a sudden small displacement parallel to its axis at the instants to be marked. While the fork is in vibration the displacements thus produced in the wavy trace on the smoked paper could readily be detected, and the interval of time between the instants at which the displacements were made would be

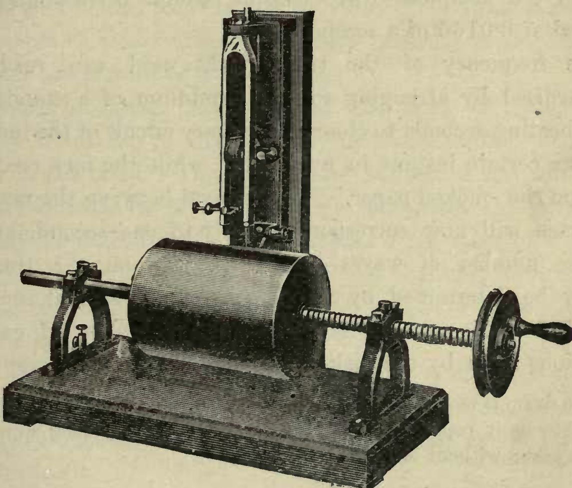


Fig. 19.—Chronograph.

given by the number of complete waves in the portion of the trace between the displacements.

The method in general use for marking the trace on a chronograph is, however, an electrical one which is simpler and more satisfactory in practice than any mechanical method. The drum and the tuning fork are connected to the secondary terminals of an induction coil, so that a spark passes between the point of the style and the drum whenever the primary circuit of the coil is made or broken. This spark produces a small white mark on the smoked paper, and in this way the trace made by the style can be marked at any instant by merely making or breaking the primary circuit.

When the frequency of the fork is high, very short intervals of time can in this way be recorded and measured. Thus, if the fork has a frequency of 500 vibrations per second, each complete wave of the trace on the smoked paper corresponds to 0·002 of a second, and a portion of the trace, estimated to contain 8·3 complete wave lengths, would correspond to an interval of 0·0166 of a second.

The frequency of the tuning-forks used can readily be standardised by arranging for the pendulum of a standardised clock beating seconds to close the primary circuit of the induction coil at a certain instant in every beat, while the fork records its trace on the smoked paper. The interval between the marks on the trace will now correspond exactly to one second, and the average number of waves in this interval on the trace can readily be determined by counting them for several successive seconds.* This number is evidently the number of complete vibrations made by the fork in one second.

* The drum is mounted on a screw as axis, so that it moves parallel to its length as it rotates. By this arrangement a very long spiral trace may be taken without any overlapping.

CHAPTER V.

MEASUREMENT OF MASS.

20. **Mass.***—The mass of a body may be defined as the quantity of matter it contains. This definition is somewhat unsatisfactory from some points of view. The limited meaning which must be attached to the words, "quantity of matter," will be better understood at a later stage.

It must be noted, however, that the mass of a body is a quantity which can be changed only by changing the quantity of matter in the body. Thus, if we add matter of any kind to a body we increase its mass, and if we take away any portion of the body we decrease its mass. Hence, if the mass of a body is found to change, it must be inferred that the body has gained or lost matter, and the quantity of matter gained or lost is measured by the change of mass.

21. **Weight.***—The fact that all bodies with which we have to deal possess **weight** is familiar to us from everyday experience. A body is said to be *heavy* or *light* according as its weight is great or small.

It will be understood later that the weight of a body is due to the attraction exerted by the earth on the body. It is a general property of matter that any two pieces of matter mutually attract each other. The very large piece of matter which makes up the earth, and the small piece of matter which makes up the body, therefore attract each other mutually, and the force with which the earth pulls the body towards it is the weight of the body.

* See also Arts. 38 and 39. Recent speculations as to the nature of mass need not here be considered.

The weight of any body may, therefore, be defined as the downward pull which the earth exerts on that body. The direction of the pull is *vertically* downwards, towards the centre of the earth, so that when a body falls freely the line along which it falls is a *vertical* line. The magnitude of the pull depends upon the mass of the body and upon its distance from the centre of the earth.

In comparing mass and weight, it will be seen that the mass of a body is essentially constant, and cannot be changed without adding to or taking from the matter in the body. The weight of the body, on the other hand, depends upon its position relative to the earth, and changes as the distance of the body from the centre of the earth changes. The weight of a body, for example, is less at the equator than at the poles, and is found to increase slightly as the latitude of the place at which it is measured increases. This shows that the weight decreases as the distance from the centre of the earth increases. For the same reason the weight of a body decreases as its height above the sea level increases.

22. **Units of Mass.**—The standard unit of mass in England is the **pound.** The pound (avoirdupois) is defined as the mass of a piece of platinum, which is preserved with other standards at the Standards Office of the Board of Trade.

Of the numerous multiples and sub-multiples of the pound we need only notice here the *ounce* and the *grain.* The ounce is the sixteenth part of a pound, and the grain is the seven-thousandth part of a pound, so that we have

1 pound = 16 ounces = 7,000 grains,

or, in the usual abbreviated notation,

1 lb. = 16 ozs. = 7,000 grs.

The standard unit from which the **gramme** of the metric system is derived is the **standard kilogramme.** This is the mass of a piece of platinum kept at the Bureau des poids et des mèsures in Paris, and known as the *kilogramme des archives.* It was originally constructed by Borda to be equal to the mass of a cubic decimetre of pure water at 4° C., but later measurements show that it is slightly greater than this mass.

The derivatives of the standard kilogramme in general use are the **gramme** and its sub-multiples the *decigramme*, *centigramme*, and *milligramme*. Thus, we have

1 kilogramme (kg.) = 1,000 grammes.
1 gramme (grm.) = 10 decigrammes = 100 centigrammes (cgs.).
= 1,000 milligrammes (mgs.).

In physical measurements it is the general practice to express all masses in grammes, or, if they are very large, in kilogrammes. Thus, we may have 896·423 grammes, or 126·432 kilogrammes.

The mass of a cubic decimetre, or 1,000 cubic centimetres of pure water at 4° C., although not exactly equal to a kilogramme or 1000 grammes, is very nearly equal to this mass. A gramme is therefore almost exactly the mass of one cubic centimetre of pure water at 4° C. The mean result of recent measurements gives the exact mass of a cubic centimetre of water at 4° C. to be 0·999955 gramme.

The relative magnitude of the pound and the gramme is given by the following equivalents.

1 kilogramme = 2·2046 pounds.
1 gramme = ·0022 pound = 15·432 grains.
1 pound = 453·59 grammes.

For ordinary purposes it is convenient to take

1 kilogramme = 2·2 pounds,

and

1 pound = 453·6 grammes.

The equivalent, 1 gramme = 15·432 grains, is easily remembered.

23. Comparison of Mass by the Process of Weighing. —If we are provided with a measuring scale derived from the standard of length and showing all necessary multiples and sub-multiples of the unit, we have no difficulty in measuring any unknown length by means of the scale. The process by which the measurement can be made is obvious, and easily understood.

If, however, we are provided with a copy of the unit of mass, and with all necessary multiples and sub-multiples of the unit, and are required to measure the mass of a given body in terms of the unit, we cannot proceed with the measurement without some understanding as to the process to be adopted in comparing the masses.

The process generally adopted in comparing any two masses is that of **weighing.** This process is based on the understanding that the weight of a body is directly proportional to its mass, and that bodies which are equal in weight are therefore equal also in mass.

The adoption of the process of weighing for the comparison of masses may be looked upon as an extension of the definition of mass. It implies that the mass of a body, or "the quantity of matter" in a body, is a quantity which is directly proportional to the force of attraction which the earth exerts on the body.

The instrument by which masses are compared by the process of weighing is called a **balance.** A simple form of balance is shown in Fig. 20.

It consists essentially of a horizontal **beam** balanced centrally on a knife-edge, and carrying a **scale-pan** suspended from each end of the beam at points equidistant from the central knife-edge. The beam is thus balanced as a horizontal lever on the central knife-edge as a fulcrum, and the scale-pans are suspended one on each side of the fulcrum, at the ends of **equal arms.**

From this construction it is obvious that if the body whose mass is to be determined is placed in one pan, and multiples and sub-multiples of the unit of mass are placed in the other pan until the beam is exactly balanced in equilibrium on its knife-edge, the weight of the mass in one pan must be exactly equal to the weight of the mass in the other pan; for these two weights act against each other on the lever, at points equidistant from the fulcrum, and can, therefore, balance each other exactly only when they are equal.

The balance thus indicates when the weights of the masses in the scale pans are equal, and it is understood as the basis of the process of weighing, that the masses are equal when their weights are equal. If, therefore, we sum up the multiples and sub-multiples of the unit of mass which must be placed in one scale

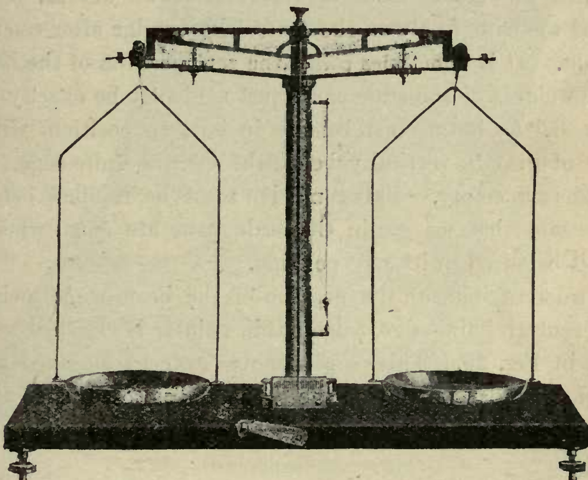


Fig. 20.—Balance.

pan to balance any given body in the other pan, we get the measure of the mass of the body, in terms of the unit of mass.

The theory of the balance, and the details of its construction, will be dealt with more fully at a later stage. It may, however, be noted here, in anticipation, that the central knife-edge, made of steel or agate, rests upon a small plate or plane of steel or agate, and that the pans are suspended from the beam by means of an inverted stirrup arranged as shown in Fig. 20, so that a small steel or agate plane in the stirrup rests on a knife-edge let into the upper edge of the beam. In this way, friction at the fulcrum is reduced to a minimum, and the length of an arm is determined definitely as the distance between two knife-edges.

In all well constructed balances the arms are exactly equal, and the three knife-edges on the beam are parallel and in the same plane. The plane should be horizontal when the beam is in such a position that its centre of gravity is vertically below the central knife-edge. This position is evidently the position in which the beam comes to rest of its own accord, if it is allowed to swing freely on the central knife-edge after removing the pans. It is sometimes called the zero position of the beam.

The weights of the scale pans must evidently be exactly equal —that is, the beam must balance in its zero position, with its centre of gravity vertically below the central knife-edge, when the pans are empty. This condition must be fulfilled before it can be said that masses in the scale pans are equal when the beam is balanced in its zero position.

In order to indicate the position of the beam when swinging on its central knife-edge, a long thin pointer is attached to it as shown in Fig. 20. This pointer moves over a scale fixed at the base of the pillar supporting the beam, and is so adjusted that it points to the zero of the scale when the beam is in its zero position.

The central pillar which supports the beam can, in most balances, be raised and lowered by means of a small eccentric cam fixed at the base of the pillar, and worked by the small handle shown at the front of the base board in Fig 20. When the pillar is raised the beam swings freely on the central knife-edge, and the balance is ready for use, but when the pillar is lowered the beam rests on supports provided for the purpose. In this way, the central knife-edge and the plane on which it rests are protected from wear when the balance is not in use.

The masses used as standards in the process of weighing are multiples and sub-multiples of the standard unit, or some convenient derivative of it. They are usually called "weights," because they are used in "weighing," and are generally arranged in sets on a definite plan. The arrangement usually adopted is indicated in Fig. 21, which shows a box containing a set of weights for weighing masses up to 100 grammes. In this set the gramme is the unit of mass, and the multiples and sub-multiples provided include:—

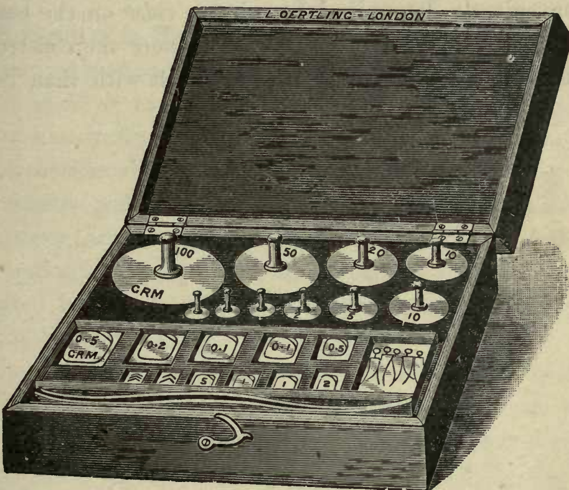


Fig. 21.—100-gramme box weights.

50,	20,	20,	10	Grammes.
5	2	2	1	
·5	·2	·2	·1	
·05	·02	·02	·01	
·005	·002	·002	·001	

or,

50,	20,	20,	10	Grammes.
5,	2,	2,	1	
5,	2,	2,	1,	Decigrammes.
5,	2,	2,	1,	Centigrammes.
5,	2,	2,	1,	Milligrammes.

It will be seen that with this set of "weights," it is easy to make up to the nearest milligramme any desired mass up to 111·111 grammes.

The gramme "weight" and all "weights" of higher denomination are generally made of brass, and are often gilt or nickel plated to preserve them. The sub-multiples of the gramme are usually made of thick sheet platinum or aluminium.

It will be seen later that the milligramme "weights" are not much used. Milligrammes, and tenths of a milligramme, are more conveniently determined by using a *rider* on the beam of the balance as explained in Chapter xv., where the construction and use of the balance are more fully dealt with than in this article.

CHAPTER VI.

VELOCITY.

24. **Motion of a Material Particle.**—Motion may be defined as *change of position;* when a body is changing its position it is said *to move* or to be in motion. The motion of a body is more conveniently studied if the motion of a material particle is first considered.

A *material particle* is a particle of matter reduced to such small dimensions that the space which it occupies is reduced practically to a point. The position of a particle may, therefore, be indicated by a point, and as the particle moves from point to point it traces out a line.

When a particle moves from one point to another the line which it follows is called a *path of motion* between the two points, and the *straight line* joining the two points is the *displacement* of the particle.

25. **Motion of an Extended Body.**—An *extended body*—that is, a body whose dimensions extend beyond those of a particle—may be supposed to be made up of an infinite number of material particles.

In what follows the relative positions of the particles of the body are supposed to be fixed, so that the size and shape of the body remain unchanged. That is, the body is supposed to be a *rigid body,* and to be, as such, incapable of any change of *configuration.*

An extended body is capable of two distinct kinds of motion.

When a body moves in such a way that any straight line in it remains parallel to its initial position throughout the motion, the motion of the body is said to be *motion of translation.*

When a body moves in such a way that it rotates round a fixed axis, and the paths of motion of all particles in it are circles round points on the axis as centres, the motion is known as *motion of rotation.*

When a body moves in any way it can be shown that its motion at any instant is compounded of a simple motion of translation and a simple motion of rotation, as defined above.

When a body is displaced from one position to another by simple translation the displacement is measured by the displacement of any particle in the body.

In the case of simple motion of rotation the displacement of a body is measured as *angular* displacement. It is given by the angle through which the line joining any point in the body to the centre of its circular path of motion, rotates.

It may be seen that the displacement of a particle or a body is a quantity which possesses direction as well as magnitude; that is, it cannot be specified completely without giving its direction as well as its magnitude.

26. **Velocity.**—When a particle is in motion it takes time in passing from point to point in its path, and a consideration of the time taken in passing over the distance between any two points introduces the idea of *time rate of motion*, or the distance passed over *per unit of time.*

The rate of motion of a particle is generally called the **velocity** of the particle.

The general unit adopted for the measurement of velocity is derived from the units of length and time, and *the unit of length per unit of time* is taken as the unit of velocity. That is, if the foot and second be taken as the units of length and time respectively, the *foot per second* is the corresponding unit of velocity; or, if the centimetre and second are the units of length and time, the *centimetre per second* is the corresponding unit of velocity.

Hence, if a particle moves over a distance s units in a time t units, the *average* magnitude of its velocity for the time considered is $\frac{s}{t}$ units of velocity.

In the same way if a particle, in a very short time τ, *taken so as to include a particular instant* in its time of motion, passes over a very short distance δ, the average magnitude of the velocity for this short interval of time is given by $\frac{\delta}{\tau}$, and may be taken (if τ is small enough) as *the velocity of the particle at the particular instant* considered.

This idea of the velocity of a particle at a particular instant in its time of motion may also be presented as the velocity of a particle at a particular point in its path of motion. Thus, if a particle passes over a very short distance δ, *taken so as to include a particular point* in its path of motion, in a very short time τ, the average value of the velocity of the particle for this very short time is given by $\frac{\delta}{\tau}$, and may be taken (if δ is small enough) as the velocity of the particle at a particular point in its path. For example, if a particle at a particular point in its path passes over a hundredth of a centimetre (taken so as to include the point) in a thousandth of a second, its velocity at that point cannot differ much from 10 cms. per second, the average magnitude of the velocity for the interval taken. It will be understood from what has been said above that if δ and τ are both infinitely small, the limiting value of the ratio $\frac{\delta}{\tau}$ when δ and τ are both infinitely small gives the velocity of the particle at the instant at which τ vanishes or at the point at which δ vanishes. The direction of the velocity of a particle at any instant is the direction in which the particle is moving at that instant.

The velocity of a particle is said to be of *uniform magnitude* when the particle passes over equal distances in equal times, no matter whether the times be long or short. That is, the velocity is of uniform magnitude if the distance passed over by the particle in any given time is directly proportional to the time. Hence, if a particle moves with a velocity of uniform magnitude v for a time t, and the space passed over by the particle be denoted by s, we have

$$s = v\,t.$$

When the velocity of a particle is not of uniform magnitude it is said to be of *variable magnitude.*

From what has been said it will be understood that velocity is a vector quantity possessing direction as well as magnitude. It is necessary, therefore, in order to specify a velocity completely to give both its magnitude and its direction. It follows, too, that the velocity of a particle is constant and invariable only if its direction as well as its magnitude remain unchanged. That is, if the velocity of a particle remains unchanged it must be of uniform magnitude, and the particle must move along a straight line. When this is the case the velocity of the particle is properly called *uniform velocity.* In any other case the velocity is *variable velocity.*

It has been proposed to use the term *speed* when velocity is considered as a magnitude only, without reference to direction. A velocity of uniform magnitude could then be spoken of as a *uniform speed.* It seems, however, unnecessary and undesirable to add to the number of terms already in use in this subject.

When the velocity of a particle is *variable* the space covered in a given time cannot be determined by the simple relation given above. If the velocity of the particle *at any instant* be denoted by v the space passed over in a *very short* time, τ, taken so as to include the given instant, differs very little from $v\tau$, and the shorter the time τ the more nearly does $v\tau$ give the space covered in that time. Hence, if a particle moves with variable velocity for a time, t, and we suppose this time to be divided into a very large number, n, of very small *equal* intervals, each denoted by τ, then when n is large enough, the space covered by the particle in the time t is given by—

$$s = v_1\tau + v_2\tau + v_3\tau \quad . \quad . \quad + v_n\tau,$$

where $v_1, v_2, v_3 \quad . \quad . \quad . \quad v_n$ denote the velocities of the particles at the middle instants of the short intervals taken in order from the first to the nth, or last.

That is, $$s = (v_1 + v_2 + v_3 \quad . \quad . \quad . \quad v_n)\tau.$$

But $$\tau = \frac{t}{n},$$

and, therefore, $$s = \frac{(v_1 + v_2 + v_3 \quad . \quad . \quad . \quad v_n)}{n} . t.$$

Now, if n is infinitely great, $\frac{v_1 + v_2 + v_3 \quad . \quad . \quad . \quad v_n}{n}$ is the *average velocity* of the particle during the time t. If this average velocity is denoted by $\bar{v}$, we have $s = \bar{v}t$.

The investigation given above may be put in a graphical form which is more easily followed and leads to a more definite result.

If a curve be plotted so that the ordinate at any point represents the velocity of the particle at the instant indicated by that point, and the abscissa represents the time of motion measured from a particular instant as starting point, the curve will show how the velocity of the particle varies from instant to instant during the time of motion.

Let CD, Fig. 22, be a *velocity curve* plotted in this way for a particular case, and consider the motion of the particle for the very short interval of time represented by *ab*. The velocity at the beginning of this interval is represented by *ac* and at the end of the interval by *bd*. If the velocity remained constant throughout the interval at the value it has at the beginning of the interval, the space described during the interval would be represented by the area of the rectangle *abce*. The number that measures the length of *ac* is the same as the number which measures the velocity of the particle at the instant indicated by the point *a*, and the number that measures the length of *ab* is the same as the number that measures the duration of the interval of time it represents; the product of these two numbers is, therefore, the number that measures the area of the rectangle *abce* contained by *ac* and *ab*, and also the space passed over by a particle moving with a uniform velocity represented by *ac* for a time represented by *ab*. This space is, therefore, represented by the area of the rectangle in the usual way; that is, *the number that measures the one is the same as the number that measures the other.*

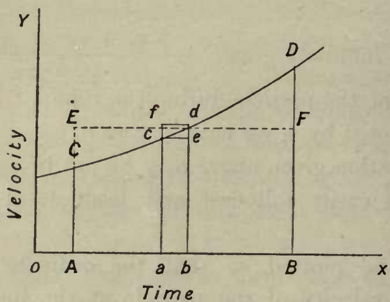


Fig. 22.

Similarly, if the velocity of the particle were the same throughout the interval as it is at the end of the interval, the space passed over during the interval would be represented by the rectangle *abdf*.

Now the space actually passed over during the interval must be less than that represented by *abdf* and greater than that represented by *abec*, and the difference between each of these extreme values and the space represented by the strip *abdc* can be made as small as we please by making *ab* small enough. Hence, when *ab* is small enough the space actually passed over by the particle in the short time represented by *ab* is represented by the area of the strip *abcd* which stands on *ab* as its base, and is bounded along *cd* by the portion of the curve intercepted between the ordinates *ac* and *bd* which form its sides.

It follows from this that if we consider the motion of the particle for any finite time represented by AB, the space passed over in that time must be represented by the area ABDC which lies between the ordinates AC and BD, and is bounded by the curve along CD. The time represented by AB may be divided into a very large number of small equal intervals similar to that represented by *ab*, and if AB be divided into a corresponding number of short equal lengths, the space passed over in each interval is represented by the area of the strip similar to *abec* which stands on the length which represents it. The total space passed over must, therefore, be represented by the area ABDC, which is made up of all the strips which stand on the short lengths into which AB is divided.

It will be seen that if AE, the height of the rectangle ABFE, shown in Fig. 22, is such that the area of the rectangle is equal to the area ABDC, then AE is the mean or average of the ordinates between A and B and represents the average velocity denoted by $\bar{v}$ above.

When a body moves with motion of translation, the velocity of the body at any instant is the velocity of any particle in it at that instant.

Since velocity is a vector quantity, velocities may be compounded or resolved by any of the rules for compounding or resolving vector quantities. The application of the parallelogram and triangle rules to the composition and resolution of velocities, gives rise to the theorems known as the *parallelogram of velocities* and the *triangle of velocities.* These theorems are merely the statement of the general rules for vectors with specific relation to velocities.

Numerical Examples.—1. A bullet is projected horizontally from the top of a tower, and as it falls its velocity at a certain instant is known to be made up of a horizontal component 80 feet per second in magnitude, and a vertical component 64 ft. per second in magnitude. Find its velocity.

Here, if we take AB, 80 units long, to represent the horizontal component, and AC, 64 units long, to represent the vertical component, and then complete the rectangle ABDC, as in Fig. 23, the diagonal AD, drawn from A to D, represents the resultant of the two components, and gives the magnitude and direction of the velocity of the bullet at the instant considered.

If the figure is drawn carefully to scale, and the length of AD measured, it will be found to be about 102·4 units long. The magnitude of the velocity is, therefore, 102·4 feet per second, and its direction is such that it makes an angle BAD with the horizontal. This angle can be measured with a protractor, and the velocity of the bullet at the instant considered, can then be fully specified.

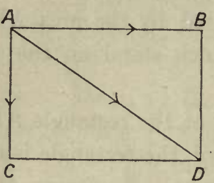


Fig. 23.

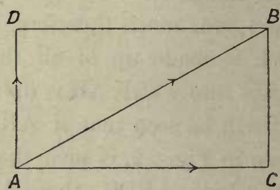


Fig. 24.

Instead of adopting a graphical method the magnitude of the velocity can be calculated by the relation, $R^2 = P^2 + Q^2$, given in Art. 9. We get

$$R^2 = 80^2 + 64^2,$$

and, therefore,

$$R = 102{\cdot}45.$$

The angle BAD can also be specified mathematically. From the figure it will be seen that $\frac{DB}{BA} = \tan BAD$; that is, $\tan BAD = \frac{64}{80} = \frac{4}{5}$; or, BAD is an angle whose tangent is $\frac{4}{5}$.

The magnitude and direction of the velocity of the bullet at the given instant are thus completely determined.

2. A bullet is fired from a rifle in a direction making an angle of 30° above the horizontal. At the instant the bullet leaves the muzzle of the rifle its velocity is 1,000 feet per second, find the magnitude of the horizontal and vertical components of this velocity.

Let AB, Fig. 24, represent the velocity of the bullet in magnitude and direction. Construct on AB as diagonal the rectangle ACBD with the side AC horizontal and the side AD vertical. The sides AC and AD now represent, respectively, the horizontal and vertical components of the velocity represented by AB.

The lengths of AC and AB can now be found in several ways.

If the figure is drawn accurately to scale the lengths can be measured directly. If this is done it will be found that AC is about 866 units long, and AD 500 units long, indicating that the horizontal component is about 866 ft. per sec., and the vertical component 500 ft. per sec.

From the geometry of the figure it is easily seen that

$$\frac{AB}{AC} = \frac{2}{\sqrt{3}}, \text{ or } AC = AB \cdot \frac{\sqrt{3}}{2} = 500\sqrt{3} = 866{\cdot}025.$$

and,

$$\frac{AB}{AD} = \frac{2}{1}, \text{ or } AD = AB \cdot \frac{1}{2} = 500.$$

That is, the horizontal component is $500\sqrt{3}$ ft. per sec., or 866·025 ft. per sec., and the vertical component is 500 ft. per sec. The same result is obtained more concisely and expeditiously if we apply the relation given in Art. 9. If P be taken to denote the horizontal component, and Q the vertical component in feet per second, we have at once

$$P = R\cos\alpha = 1{,}000 \cdot \cos 30 = 1{,}000 \cdot \frac{\sqrt{3}}{2} = 500\sqrt{3},$$

and

$$Q = R\sin\alpha = 1{,}000 \cdot \sin 30 = 1{,}000 \times \frac{1}{2} = 500.$$

That is, the horizontal component is $500\sqrt{3}$ ft. per sec., and the vertical component is 500 ft. per sec.

27. **Relative Velocity.**—The velocity of a point B relative to a point A is the rate at which the point B changes its position relative to A.

If the two points are in motion with the same velocity there can be no change in their relative position, and the velocity of one relative to the other is zero.

If, however, the points are in motion with different velocities, the velocity of one relative to the other depends upon the magnitude and direction of their individual velocities.

Let the velocity of the point A in Fig. 25 be represented in magnitude and direction by AP, and the velocity of the point B by BQ. Now, the velocity of B relative to A will not be affected if we impress *the same velocity* on the two points. Imagine, therefore, a velocity equal and opposite to the velocity of A to be impressed on each point, and let this velocity be represented in magnitude and direction by AR for the point A, and by BR for the point B. It will be seen from the figure that the result of this is to reduce the point A to rest, and to give the point B a velocity compounded of the velocities repre-

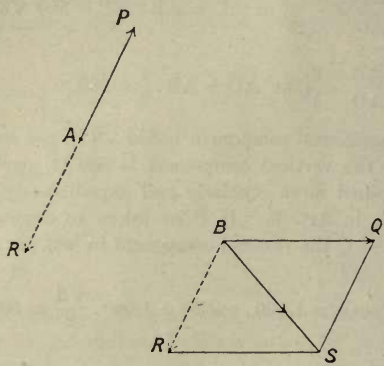


Fig. 25.

sented by BR and BQ. This velocity of B is represented in the figure by BS, and is evidently the velocity of B relative to A, for the point A is now at rest.

That is, the velocity of a point B relative to another point A is a velocity compounded of B's actual velocity and a velocity equal and opposite in direction to A's velocity.

It will be seen at once by drawing a figure that the velocity of A relative to B is equal and opposite in direction to the velocity of B relative to A.

If the points A and B move along the same line, and if velocity is considered to be positive for one direction along the line, and negative for the opposite direction, the velocity of B relative to A is evidently obtained by adding A's velocity with its sign changed to B's velocity—that is, by subtracting A's velocity from B's velocity.

What has been said above with reference to the relative velocity of two *points* applies also to the relative velocity of any two bodies moving with motion of translation.

Examples.—1. Two bodies A and B move along the same straight line with uniform velocities; the velocity of A is 20 ft. per sec., and the velocity of B is 15 ft. per sec., find the velocity of B relative to A when the bodies move (1) in the same direction, (2) in opposite directions.

Here if we take velocity in the direction in which A moves to be positive we get the following results:—

(1) The velocity of B relative to A, when A and B are moving in the same direction is given by $(15 - 20)$ ft. per sec. or -5 ft. per sec. That is, B's velocity relative to A is 5 ft. per sec. in a direction opposite to that in which A (and in this case B also) is moving. This means that if B is in front of A, B is getting nearer to A at the rate of 5 ft. per sec., or if A is in front of B, B is getting further away from A at the rate of 5 ft. per sec.

(2) The velocity of B relative to A when A and B are moving in opposite directions is given by $(-15 - 20)$ ft. per sec. or -35 ft. per sec. That is B's velocity relative to A is 35 ft. per sec. in a direction opposite to that in which A is moving.

If we suppose the two bodies A and B to be two trains, the velocity of B relative to A obtained as above is the velocity which the train B appears to have to a passenger on A who looks only at the train B.

2. A steamer, A, travelling due north at a speed of 15 knots passes another steamer, B, travelling due east at a speed of 20 knots. Find the velocity of the steamer B relative to the steamer A.

The velocity of the steamer B relative to the steamer A is the velocity obtained by compounding B's velocity with a velocity equal and opposite to that of A.

Hence, if BE in Fig. 26 represents B's velocity in magnitude and direction, and BS similarly represents a velocity equal and opposite to A's velocity. BR will, by the parallelogram of velocities, represent the velocity of B relative to A.

Since BE and BS are respectively 20 units and 15 units in length, and the angle EBS is a right angle, it follows that $BR^2 = BE^2 + BS^2$.

That is,

$$BR^2 = 20^2 + 15^2 = 625$$

or

$$BR = 25.$$

The velocity of B relative to A is therefore 25 knots in the direction represented by BR. This direction lies between S.E. and E.S.E., making an angle of nearly 37° with BE.

That is, to a passenger on board the steamer A, the steamer B appears to move away from him in a direction between SE and ESE with a speed of 25 knots.

It will be understood that all the velocities with which we

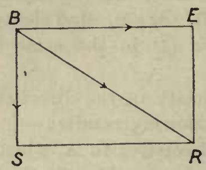


Fig. 26.

have to deal are relative velocities. When we speak of the velocity of a body we mean its velocity relative to some point at rest on the earth's surface. Motion and rest are in fact relative terms, and the use of either term implies the existence of some point of reference expressed or understood. We cannot specify the absolute position of a point in space, and cannot therefore attach any real meaning to the terms "absolute velocity" and "absolute rest" which are sometimes used.

CHAPTER VII.

ACCELERATION.

28. **Acceleration.**—When the velocity of a particle changes, the time in which any given change takes place depends upon the time rate at which the velocity changes from instant to instant.

The *rate of change* of *velocity*, or the change of velocity *per unit* of *time* is called **acceleration.** The **unit of acceleration** is unit change of velocity per unit of time. Thus, if a foot-per-second is the unit of velocity, a change of velocity at the rate of a foot-per-second *per second* is the corresponding unit of acceleration. Similarly, if a centimetre-per-second is the unit of velocity a change of velocity at the rate of a centimetre-per-second *per second* is the corresponding unit of acceleration.

Acceleration as here defined evidently applies to rate of *decrease* of velocity, as well as to rate of *increase* of velocity. Sometimes the term acceleration is limited to the rate of increase of velocity, and the term *retardation* is adopted for rate of decrease of velocity. It is, however, more convenient to use only the one term—acceleration, and to consider retardation as negative acceleration.

In determining change of velocity it must be remembered that velocity is a vector quantity, and that a change of velocity may involve a change of direction as well as a change in magnitude. Change of velocity must, therefore, be found by applying the rule for finding the difference of two vector quantities.

It is convenient, however, to consider separately the two cases in which change of velocity involves (1) a change in magnitude without change of direction, and (2) a change in direction with or without a change of magnitude.

The **first case**, in which the velocity of a particle changes in magnitude without changing in direction, is evidently a special case in which the particle moves without change of direction *along a straight line.* In this case the velocity at any instant is always in the same direction, and the difference of the velocities at the beginning and end of any interval of time gives the change of velocity in the time. Hence if u denote the velocity of the particle at any instant, and v the velocity at an instant t units of time later, *the change of velocity* in the time t is given by $(v - u)$, and the *average rate of change* of velocity, or *average change of velocity per unit of time* during this time is given by $\frac{v - u}{t}$. That is, $\frac{v - u}{t}$ is the *average acceleration* during the time t.

In the same way, it follows that if the velocity of the particle at any instant is u, and it changes in a *very short time* τ to u', then $\frac{u' - u}{\tau}$ is the average acceleration for the very short interval of time τ, and if τ is small enough the value of $\frac{u' - u}{\tau}$ may be taken as the acceleration of the particle at the particular instant considered. For example, if the velocity of a particle at any instant is 10 cms. per second, and a thousandth of a second later it is 10·01 cms. per second, the change of velocity in ·001 second is ·01 cm. per second, and the average acceleration during this short interval is $\frac{\cdot 01}{\cdot 001}$ or 10 cms.-per-second *per second.* Now, the acceleration at the instant first considered cannot differ much from this value, for the interval of time taken is too short to allow of much change, and it is clear that the shorter the interval is, the more exactly will the average acceleration for the interval give the acceleration at the particular instant considered.

The acceleration of a particle, in this case, is said to be of *uniform magnitude* when equal changes of velocity take place in equal times, however long or short the times may be. That is, the acceleration is of uniform magnitude when the change of velocity in any time is directly proportional to the time. Hence, if a particle moving in a straight line is subject to an acceleration of uniform magnitude a, the change of velocity which takes place in a time t is given by at. That is, if the velocities at the beginning and end of a time t are denoted by u and v respectively, we have

$$v - u = at.$$

or,

$$v = u + at.$$

In this case an acceleration of uniform magnitude may be said to be a *uniform acceleration*, for, since there is no change of direction to be considered, the acceleration is constant and invariable *in magnitude and direction.*

The *direction* of the acceleration in this case must evidently be along the line of motion, and will be in the same direction as the motion of the particle, or in the opposite direction, according as the motion is accelerated or retarded. If the motion is accelerated the velocity is increased, and the acceleration is positive in sign, but if the motion is retarded the velocity is decreased, and the acceleration is negative in sign.

The **second case,** in which the velocity of a particle changes in direction with or without a change of magnitude, is evidently the general case in which the particle moves along any line. In this case the direction of the velocity changes from instant to instant, and the difference of the velocities at any two given instants must be determined by the rule for vectors. It will be seen, too, that the acceleration may also change in direction from instant to instant, so that we cannot find the average acceleration for any interval of time by dividing the change of velocity in that time by the time. As a general rule, we can deal only, in this case, with acceleration at an instant. Thus, in Fig. 27, let AB represent the velocity of the particle, in magnitude and direction, at a particular instant, and let AC represent its velocity, similarly, at an instant a very short time later, then by the triangle rule given in Art. 8, BC represents the change in velocity during this very short interval of time. If, now, the magnitude of this change of velocity be denoted by δ, and the very short interval of time in which it takes place by τ, then δ/τ denotes the average acceleration for this short interval of time, and may be taken, if τ is small enough, to give the acceleration of the particle at the instant when the velocity of the particle is represented by AB. The direction of this

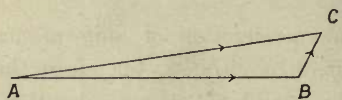


Fig. 27.

acceleration is then represented by the direction of BC. The acceleration at any instant, obtained in this way, may evidently vary in magnitude and in direction from instant to instant; that is, the acceleration is, in general, *variable*. If, however, the magnitude remains constant, the acceleration is said to be of *uniform magnitude*, and if both magnitude and direction are constant and invariable from instant to instant, the acceleration is called *uniform acceleration*, as in the case considered above. A particle which is not moving in a straight line may evidently be subject to uniform acceleration, for this merely means that the velocity of the particle is changing at a uniform rate in a constant direction which is not at any instant the same as that in which the particle is moving.

It should be noted that acceleration possesses direction as well as magnitude, and is, therefore, a vector quantity. Accelerations may therefore be compounded and resolved by the usual rules for vector quantities. The theorems known as the *parallelogram of accelerations*, *the triangle of accelerations*, and *the polygon of accelerations* deal merely with the application of these general rules to the special case of accelerations.

29. **Uniformly Accelerated Motion in a Straight Line.**—The motion of a particle along a straight line with uniform acceleration is a case of special importance.

If a particle starts from rest and moves for a time t subject to a uniform acceleration a, it evidently gains a units of velocity every unit of time during the motion, and, therefore, acquires a velocity at at the end of the time. That is, if v denote its *final velocity*, or its velocity at the end of the time considered, we have

$$v = at.$$

Similarly, when a particle moves with uniform acceleration a along a straight line, and we consider its motion for any interval of time t during the motion, it will be seen that if u denote its *initial velocity*, or its velocity at the beginning of the time, and v its *final velocity*, or its velocity at the end of the time, we have

$$v = u + at,$$

for the velocity is increased by a units of velocity every unit of time, and, therefore, gains at units in the time t.

It will be seen, too, that since the velocity of a particle moving with uniform acceleration in a straight line *changes uniformly with time*, its mean or average value for any interval of time is the arithmetic mean of its initial and final values for the interval considered, and is equal, also, to the actual velocity of the particle at the middle of the interval. Thus, let u denote the initial velocity of the particle at the beginning of the time t, and imagine this time to be divided into n equal intervals, each equal to $\frac{t}{n}$; then, if a denote the uniform acceleration to which the particle is subject, the successive velocities of the particle at successive instants, taken from the beginning of the time at equal intervals, each equal to $\frac{t}{n}$, are given by

$$u, u + a\frac{t}{n}, u + a\frac{2t}{n}, u + a\frac{3t}{n} \quad . \quad . \quad . \quad u + at.$$

These velocities form an arithmetical progression of $(n + 1)$ terms, however large n may be, and their sum by the usual algebraic rule for the summation of a series in arithmetical progression is $\{u + (u + at)\}\frac{n+1}{2}$. The mean or average of these velocities is, therefore, $\frac{u + (u + at)}{2}$, *and if n is supposed to be infinitely great,* this is evidently the mean or average velocity of the particle for the time t in the sense explained in Art. 26.

The mean or average velocity of the particle for the time t is thus the arithmetic mean of u, and $(u + at)$ the initial and final velocities of the particle for the time, and its value is, therefore, equal to $u + \frac{at}{2}$, which is the actual velocity of the particle at the middle of the time.

The space passed over by the particle in any given interval of time can now be readily found. For if $\bar{v}$ denote the average velocity of the particle for any time t, then the space or distance passed over in this time is given by $s = \bar{v}t$ as explained in Art. 26.

If the particle starts from rest and moves with uniform acceleration a in a straight line for a time t, the average velocity of the particle for the given time is $\frac{1}{2}\,at$, and the space or distance passed over in this time is given by

$$s = (\tfrac{1}{2}\,at)t,$$

or

$$s = \tfrac{1}{2}\,at^2.$$

Similarly, if the particle at any instant has an initial velocity

u, and moves in a straight line with uniform acceleration a for a time t, its average velocity for the time is $(u + \frac{1}{2} at)$, and the space passed over in the time is given by

$$s = (u + \tfrac{1}{2} at)t,$$

or

$$s = ut + \tfrac{1}{2} at^2.$$

The graphical method explained in Art. 26 can easily be applied in this case. The velocity increases uniformly with the time of motion so that the curve showing how the velocity varies with the time is a straight line. If the time of motion is measured from rest, the straight line passes through the origin as shown in Fig. 28, where OV represents this *velocity curve.* From this figure it will be seen that if OA represents a time t

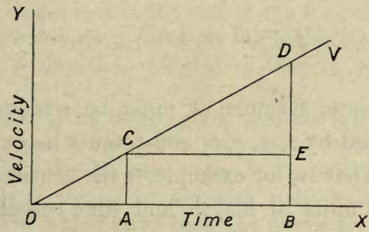


Fig 28.

from rest, the space passed over in this time is represented by the area of the triangle OAC—that is, by $\frac{1}{2}$ OA . AC. Now OA represents the time t and AC represents the velocity, at, acquired by the particle in the time t, so that the space passed over in the time t from rest is given by

$$s = \tfrac{1}{2} \,.\, t \,.\, at, \text{ or } s = \tfrac{1}{2} at^2.$$

Similarly, it can be seen that the space passed over in any interval of time t, represented by AB, is represented by the area ABDC. Now, if CE be drawn parallel to AB and cutting BD in E, it can be seen from the figure that

$$\text{ABDC} = \text{ABEC} + \text{CED},$$

and the area of $\text{ABDC} = \text{AC} \,.\, \text{AB} + \tfrac{1}{2} \text{AB} \,.\, \text{DE}.$

In this relation the area of ABDC represents the space passed over in the time t, AC represents the initial velocity of the particle at the beginning of the time t, and DE represents the additional velocity, at, acquired by the particle in the time t. Hence, it follows that

$$s = ut + \tfrac{1}{2} . t . at,$$

or
$$s = ut + \tfrac{1}{2} at^2.$$

The same result might have been obtained by writing

The area ABDC $= \dfrac{AC + BD}{2} . AB$, which indicates that

$$s = \frac{u + (u + at)}{2} . t,$$

or
$$s = ut + \tfrac{1}{2} at^2,$$

as before.

In applying these formulæ it must be remembered that the quantities denoted by s, u, v, a, and t, must be expressed in consistent units. That is, for example, if the centimetre and second are taken as the units of length and time respectively, the unit of velocity must be the centimetre-per-second and the unit of acceleration the centimetre-per-second per second. It must be remembered, too, that the sign of a must be taken positive or negative according as the motion is accelerated or retarded. Or, more generally, if the direction in which the particle is moving along the line at any instant be taken as the positive direction, the opposite direction along the line must be taken as the negative direction, and this sign convention must be taken to apply to all the quantities involved in any formula.

Numerical Example.—For example, if a particle moving in a straight line subject to a uniform acceleration of 20 cms. per sec. per sec. in a direction opposed to that in which it is moving, has, at a particular instant, a velocity of 45 cms. per sec., find (a) the velocity of the particle 5 seconds later, and (b) the distance passed over in this interval of 5 seconds.

(*a*) Here, by applying the relation,

$$v = u + at,$$

we get $$v = 45 + (-20)\,5\,;$$

or $$v = 45 - 100 = -55.$$

That is, the particle at the end of the given interval of 5 seconds is moving with a velocity of 55 cms. per sec. in a direction opposite to that in which it was moving at the instant first considered.

(*b*) Here, by applying the relation,

$$s = ut + \tfrac{1}{2}\,at^2,$$

we get $$s = 45 \times 5 + \tfrac{1}{2}\,(-20)\,25\,;$$

or $$s = 225 - 250 = -25.$$

This result means that at the end of the 5 seconds the particle is 25 cms. from its starting point (at the beginning of the 5 seconds) measured in a direction opposite to that in which it was then moving.

In the formulæ given above s gives the space or distance passed over so long as the particle moves in the same direction throughout the time considered. In this case, however, the particle evidently moves for $2\frac{1}{4}$ seconds in the initial direction until its initial velocity is reduced to zero; it then turns back and moves for $2\frac{3}{4}$ seconds in the opposite direction. It will be found, by working out the results, that during the first interval of $2\frac{1}{4}$ seconds the particle moves over 50·625 cms. in the positive direction, and during the remaining $2\frac{3}{4}$ seconds it moves over 75·625 cms. in the negative direction, so that at the end of the 5 seconds it is (75·625 − 50·625) cms., or 25 cms. from the starting point in the negative direction.

It is sometimes convenient to eliminate t from the formulæ

$$s = \tfrac{1}{2}\,at^2 \text{ and } v = at$$

and also from

$$s = ut \times \tfrac{1}{2}\,at^2 \text{ and } v = u + at.$$

In the first case we get

$$v^2 = a^2t^2 = 2\,a\,.\,\tfrac{1}{2}\,at^2 = 2\,as,$$

or $$v^2 = 2\,as\,;$$

and in the second case,

$$v^2 = u^2 + 2\,uat + a^2t^2\,;$$

or $$v^2 = u^2 + 2\,a\,(ut + \tfrac{1}{2}\,at^2)\,;$$

or $$v^2 = u^2 + 2\,as.$$

These results are useful when the relation between u, v, a, and s is required.

When a particle moves *from rest* with uniform acceleration in a straight line the characteristics of the motion are concisely expressed by the two formulæ—

$$v = at, \quad . \quad . \quad . \quad . \quad (1)$$

$$s = \tfrac{1}{2}\,at^2, \quad . \quad . \quad . \quad (2)$$

which are given above.

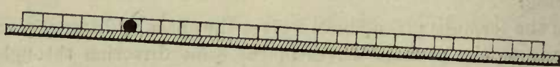


Fig. 29.

In these formulæ a is a constant, and they, therefore, indicate

(1) That the velocity acquired by the particle during any time measured from the instant of starting is *directly proportional to the time.*

(2) That the space passed over by the particle in any time measured from the instant of starting is *directly proportional to the square of the time.*

The motion of a particle moving with uniform acceleration in a straight line may be studied experimently by the method of the following experiment:—

Experiment 1.—Set up a long inclined plane (with a V-groove cut along its length) at a small angle with the horizontal, so that a large steel bearing ball will roll slowly down the plane along the groove.

A scale should be fixed on the plane, parallel to the groove, as shown in Fig. 29, so that the position of the ball can be read at any instant during its motion down the plane. The motion of the ball down the plane is a rolling motion compounded of motion of rotation and motion of translation, but the particle at the centre of the ball evidently moves in a straight line down the plane. Now adjust a metronome to tick half seconds or use a clock which ticks half seconds distinctly, and follow the motion of the ball down the plane in the following manner.

Let the ball rest in the groove, supported by a small block of wood, with its centre opposite the zero of the scale. At a particular tick of the clock or metronome remove the block suddenly away from the ball so as to let the ball begin at this instant to roll down the plane. Then follow the motion of the ball down the plane, and at every

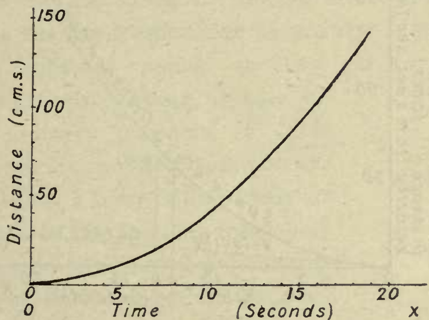


Fig. 30.

successive tick read off its position on the scale. If this operation is repeated several times it will be found that, with practice, perfectly consistent readings may be obtained. The inclination of the plane to the horizontal should be small, in order that the motion of the ball may be slow.

Record the readings obtained in this way and then plot from the readings a smooth curve, showing how the distance travelled by the ball from rest varies with the time of motion.

A curve similiar to that shown in Fig. 30 will be obtained. If this curve is examined it will be found that the ordinate at any point representing the distance travelled from rest, is proportional to the square of the abscissa representing the time of motion. That is, the distance travelled from rest is directly proportional to the square of the time of motion, and the centre of the ball moves down the plane with uniform acceleration in a straight line. If we now apply the relation, $s = \frac{1}{2} at^2$, the value of a can be determined by substituting in the relation values of s and t obtained from the co-ordinates of any suitable point on the curve.

The velocity of the centre of the ball at any instant, can also be deduced from this curve. Let ab, in Fig. 31, represent any very short interval of time during the motion of the ball down the plane.

Then de, the difference of the ordinates ac and bd, evidently represents the distance passed over in the short time represented by ab. Hence if we divide the distance represented by de by the time represented by ab, we get the average velocity of the ball during the short interval of time represented by ab, and this average velocity may be taken as the actual velocity at the middle of the interval.

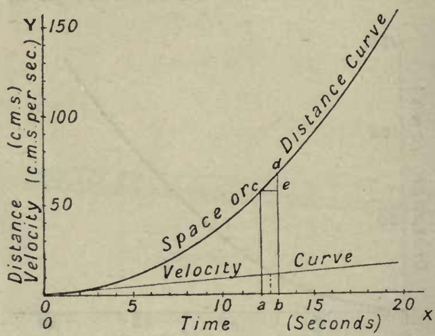


Fig. 31.

If therefore we divide the whole time of motion into a number of consecutive short intervals similar to that represented by ab, and plot at the middle point of the short length representing each interval, an ordinate representing the velocity at the middle of the interval, as obtained above, we obtain a curve which shows how the velocity of the ball varies with the time of motion.

The curve obtained in this way is known as the *velocity curve*. In this case it is a straight line, as shown in Fig. 31. This means that the ordinate at any point on the line is directly proportional to the abscissa, and indicates, therefore, that the velocity of the ball at any instant is directly proportional to the time of motion from rest. If we apply the relation $v = at$ to this line the value of a is readily found by substituting in the relation the values of v and t given by the co-ordinates of any point on the line, and then calculating out the value of a. This value should agree with the value obtained from the space curve by the help of the relation $s = \frac{1}{2} at^2$.

In plotting the velocity curve it will generally be found sufficient to divide the time of motion into half second intervals, and it will be necessary to plot the ordinates on a larger scale than those of the space curve.

30. **Acceleration due to Gravity.**—When a body falls freely from rest it falls vertically in a straight line, and it is found that the distance travelled from rest in any time is directly proportional to the square of the time. This shows, that a body in falling is subject to uniform acceleration directed vertically downwards along the line of fall. This acceleration is known as **the acceleration due to gravity.** Its value differs slightly at different points on the earth's surface, but for places in Great Britain it may be taken for ordinary purposes, as 32·2 feet-per-sec. per sec., or 981·2 cms.-per-sec. per sec. That is, a body falling freely in vacuo (so as to be free from air resistance) is subject to a uniform increase of velocity at the rate of 32·2 feet-per-sec. per sec., or 981 centimetres-per-sec. per sec. Similarly, if a body is projected vertically upwards it is subject to a *decrease* of velocity at the same rate. The direction of the acceleration due to gravity is vertically downward, and so causes acceleration of downward motion and the retardation of upward motion.

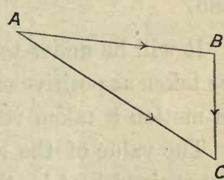


Fig. 32.

When a body is projected horizontally or in any direction it is still subject to the acceleration due to gravity—that is, the change in its velocity from instant to instant, as explained in Art. 28, always takes place in *the same direction and at the same rate.* Hence, if the velocity of the body can be represented at a given instant by AB (Fig. 32), and by AC at an instant t units of time later, then BC, which represents the change of velocity in the time t, will be directed vertically downwards, and if this change of velocity be denoted by v, then $\frac{v}{t}$ will give the acceleration* due to gravity.

In the same way, if the velocity of the body at any instant is represented by AB, the velocity at any instant t units of time later is given by compounding with the velocity represented by AB, a velocity represented by BC, taken in a vertical direction and equal to gt, where g denotes the acceleration due to gravity.

The acceleration due to gravity is usually denoted by g, and the formulæ of the foregoing article, when applied to the motion of bodies subject to this acceleration, are usually written with g instead of a, so that we have—

$$v = gt, \text{ or } v = u + gt,$$

and,

$$s = \tfrac{1}{2} gt^2, \text{ or } s = ut + \tfrac{1}{2} gt^2.$$

It will be understood in applying these formulæ that g must be taken as positive or negative according as the positive direction of motion is taken vertically downwards or vertically upwards.

The value of the acceleration due to gravity at any place is determined best by the pendulum method explained in Art. 34. It can, however, be determined roughly by direct methods. The chronograph described in Art. 19 may, for example, be adjusted to record electrically the time taken by a suitable body, such as a steel ball or a bullet, in falling from rest through a known distance. Then if s denote the distance, and t the time recorded by the chronograph, we have $s = \frac{1}{2} gt^2$, and from this relation g can be determined.

A common form of this method in which the time record is traced by the tuning fork on the falling body itself is described below.

* It should be noted that in this case $\frac{v}{t}$ gives the acceleration whether t be large or small, because the acceleration is uniform in magnitude and direction. See Art. 28.

Experiment 2.—Arrange, as shown in Fig. 33, that a long strip of plate glass, G, falls vertically between guides in such a way that as it falls past the end of the tuning fork T, a short bristle on the end of one of the prongs traces, when the fork is in vibration, the usual wavy trace on the smoked surface of the glass.

The trace need not be taken from the starting point of the strip's fall. It is better to let the strip fall from a point as far as possible above the fork, and to let the trace be taken at the end of the fall where the strip is moving very rapidly. An indiarubber pad should be arranged to receive the strip at the end of its fall.

The acceleration due to gravity can now be calculated from data given by the trace on the strip of glass in the following way.

Measure off on the trace the lengths of two *consecutive portions* each containing the *same number* of complete waves. These lengths evidently give the distances through which the plate falls in two consecutive equal intervals of time.

Let d_1 and d_2 denote these distances, and let t denote the duration of the equal intervals of time. Then, d_1/t is the average velocity of the plate during the first interval, *and gives, therefore, the actual velocity of the plate at the middle of this interval.* Similarly, d_2/t gives the actual velocity of the plate at the middle of the second interval.

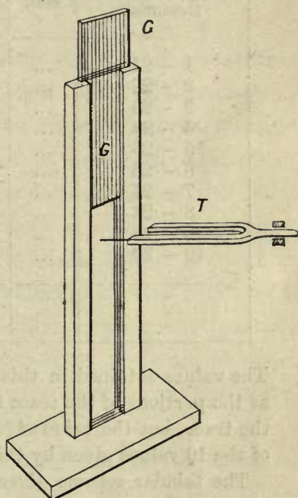


Fig. 33.

A *change of velocity*, $\frac{d_2 - d_1}{t}$, therefore, takes place in the interval between the middle of the first interval and the middle of the second interval; this interval is itself equal to t, and *the rate of change of velocity*, or the acceleration of the plate is, therefore, given by $\frac{d_2 - d_1}{t^2}$. Now, t is equal to p/n seconds, if p denote the number of complete waves in each of the measured portions of the trace, and n the number of vibrations per second made by the fork.

We, therefore, have

$$g = \frac{(d_2 - d_1)\, n^2}{p^2}.$$

The best method of getting an accurate result from a good trace by this method of calculation is as follows.

Number the wave lengths on the trace from crest to crest, from 1 to 50, using a fine needle point to write the numbers on the smoked surface. Then find a number of values of d_1 and d_2 for intervals corresponding to the time of twenty vibrations of the fork, by making the measurements indicated in the table given below.

d_1.		d_2.		$d_2 - d_1$.
Portion of Trace Measured.	cms.	Portion of Trace Measured.	cms.	cms.
1 – 21	...	21 – 41	...	...
2 – 22	...	22 – 42	...	...
3 – 23	...	23 – 43	...	...
4 – 24	...	24 – 44	...	...
5 – 25	...	25 – 45	...	...
6 – 26	...	26 – 46	...	...
7 – 27	...	27 – 47	...	...
8 – 28	...	28 – 48	...	...
9 – 29	...	29 – 49	...	...
10 – 30	...	30 – 50	...	...
			Mean.	

The values obtained in this way for d_1 and d_2 change progressively as the portions of the trace from which they are taken advance along the trace, but the value of $(d_2 - d_1)$ should be constant, and the mean of the 10 values given by this table should be fairly free from error.

The tabular scheme given above is arranged to give the mean of 10 values of $(d_2 - d_1)$ when the measured portions of the trace contain 20 complete wave lengths; it may obviously be modified in these particulars as may be required.

The value of the acceleration due to gravity may be determined with fair accuracy by methods of this kind, but the only accurate method is the pendulum method referred to above.

CHAPTER VIII.

CIRCULAR MOTION AND SIMPLE HARMONIC MOTION.

31. **Angular Velocity.**—In considering the motion of rotation of a body, the body is supposed to be *rigid*—that is, the particles which make up the body are supposed to be fixed in position relative to each other, and not subject to any relative displacement such as might be produced by any deformation of the body.

When a rigid body rotates round any straight line as axis, every particle in the body moves in a circle round a point in the axis as centre, and the **angular velocity** of the rotation is measured by the angle which the radius joining any particle to the centre of its path of motion, describes per unit of time. This angle is evidently the same for every particle in the body, and is always expressed in circular measure.

Hence if the angular velocity of a rotating body is denoted by ω, the angle which the radius joining any particle at a distance r from the axis, describes in a very short time τ, is $\omega\tau$, and since this angle is expressed in circular measure, the arc over which the particle moves is given by $r\omega\tau$ or $(r\omega)\tau$. That is, the particle moves over a distance $(r\omega)\tau$ in a very short time τ, and its *linear velocity* is therefore given by $r\omega$, where r denotes the distance of the particle from the axis of rotation, and ω denotes the angular velocity of the body. The linear velocity of any particle in the body is thus directly proportional to its distance from the axis of rotation.

If a rotating body rotates through an angle θ (circular measure) in a time t, then θ/t gives the *average* angular velocity for the time t.

If at any instant the body rotates through a very small angle δ, in a very short time τ (taken to include the instant considered), then δ/τ gives the angular velocity of the body *at the instant considered.*

If the body rotates through equal angles in equal times, however long or short the times may be, the angular velocity is *uniform,* and the angle described by the body in any time t is given by $\theta = \omega t$, where ω denotes the uniform angular velocity of the body.

Angular acceleration bears the same relation to angular velocity as linear acceleration bears to linear velocity. It may be defined as the *rate* of change of angular velocity, or the change of angular velocity *per unit of time.*

32. **Motion in a Circle.**—When a particle moves round a circle with a velocity of uniform magnitude it is subject to acceleration, for, although the magnitude of the velocity is constant, its direction changes continuously from point to point on the circle.

Imagine a particle to move round the circle PQ, Fig. 34, with a velocity of uniform magnitude v, and suppose it to move over any very short arc in a very short time τ. The direction of the velocity of the particle at the point is along the tangent to the circle at that point. Hence, if PR represent the direction of the tangent at P, and QS the direction of the tangent at Q, the velocity of the particle is, at P, along PR, and at Q, along QS, so that the direction of its velocity changes through the very small angle POQ in the very small time τ, in which it passes from P to Q.

In Fig. 35, let the velocity of the particle be represented in magnitude and direction by AB at the point P, and by AC at the point Q. The change of velocity which takes place in the very short time τ, as the particle moves from P to Q, will then be represented in magnitude and direction by BC as explained in Art. 28.

Since AB and AC are equal, each being v units in length, and the angle BAC, being equal to the angle POQ, is very small, the line BC is practically coincident with the arc of a circle described with centre A and radius AB or AC.

Hence, if the angle BAC be denoted by a, in circular measure, the length of BC is measured by va, and the change of velocity represented by BC is also measured by va. That is, the change of velocity which takes place as the particle moves from P to Q, in the time τ, is measured by va, and the average

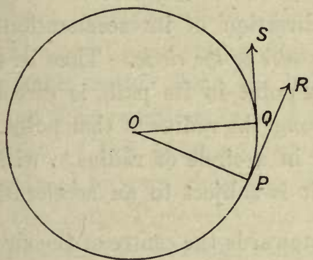


Fig. 34.

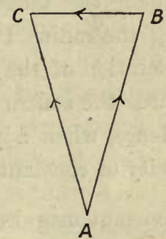


Fig. 35.

magnitude of the acceleration for this short interval of time is measured by $\frac{va}{\tau}$.

Now the angle POQ, in Fig. 34, is equal to the angle BAC, which is denoted by a, and it will be seen that $\frac{a}{\tau}$ is the angular velocity of the particle round O, and is therefore equal to $\frac{v}{r}$ where r denotes the radius of the circle. The average acceleration of the particle for the time τ is therefore given by $\frac{v^2}{r}$, for

$$v \cdot \frac{a}{\tau} = v \cdot \frac{v}{r} = \frac{v^2}{r}.$$

This is the average value of the acceleration of the particle for *any* very small interval of time, so that if the interval of time be assumed to be infinitely small, it gives the value of the acceleration at any instant during the motion of the particle. The acceleration of the particle is thus of constant magnitude for since v and r are both constant $\frac{v^2}{r}$ must be constant.

The direction of this acceleration can also be determined from Fig. 35. It will be seen that when the interval of time τ is infinitely small, the angle BAC is infinitely small, and the direction of BC is at right angles to AB and AC. This indicates that when the particle is at P, and the direction of its velocity is along the tangent PR, the direction of its acceleration is along the radius PO *towards the centre of the circle.* That is, the acceleration of the particle, at any point in its path, is directed towards the centre of the circle along the radius at that point.

Hence, when a particle moves in a circle of radius r, with a velocity of constant magnitude v, it is subject to an acceleration of constant magnitude $\frac{v^2}{r}$ directed towards the centre of the circle, at all points in its path.

33. **Simple Harmonic Motion.**—Let P, Fig. 36, be any point on the circumference of the circle APB, and AB *any* diameter of the circle. From P draw Pp perpendicular to the diameter AB and meeting it at the point p. This point p is the projection of the point P on the diameter AB. For different positions of the point P on the circumference of the circle, the point p will have different positions on the diameter AB, for the point p will in all positions be the foot of the perpendicular from P on to the diameter.

Now, imagine the point P to move round the circumference of the circle with uniform speed, and consider the corresponding motion of the point p along the diameter AB. It will be seen that as P moves round and round the circle the point p moves backwards and forwards along the diameter AB, making a complete backward and forward movement for each complete revolution made by P. The point P makes a complete revolution from any starting point on the circumference of the circle every time it passes through the starting point; the point p, therefore, makes a complete backward and forward movement from any starting point on the diameter every time it passes through the starting point in the same direction as it had at the instant of starting. Thus when P moves round the circle from B through Q, A, and Q′ back to B, the point p moves along the diameter AB from B, through O to A, and back through O to B. Or, as P moves from Q, through A, Q′ and B, back to Q, the point p moves from O to A, back through O to B, and then back to O again.

The point p, moving in this way, is said to move with **simple harmonic motion** along the line AB. That is, if a point moves round the circumference of a circle with uniform speed, the projection of this point on any diameter of the circle moves backward and forward along the diameter in simple harmonic motion.

The point P moves round the circle with uniform speed, and, therefore, describes each complete revolution in the same time. The point p makes a complete backward and forward movement for each complete revolution made by P, and must, therefore, describe each complete movement in a definite constant period of time equal to the time occupied by P in making one complete revolution. This period of time is known as the **period of the motion.** In the case of a point moving in simple harmonic motion the period of the motion may, therefore, be defined as the time occupied by the point in making one complete backward and forward movement.

The line AB along which the point p moves in simple harmonic motion is the path of the motion, and O is the middle point or centre of this path. The distance OA or OB is, therefore, the greatest distance the point travels from O during its motion. This distance is called the **amplitude** of the motion.

The distance of a point in simple harmonic motion from the centre of its path is sometimes called the **displacement** of the point. If this term is used, the amplitude of the motion may be defined as the maximum displacement of the point during the motion.

The displacement at any instant of a point in simple harmonic motion may be expressed in terms of the period and amplitude of the motion. Thus, in Fig. 36, if ω denote angular velocity of the point P round O, and if we suppose the point P to start from Q, and to take a time t to travel from Q to P, the angle POQ will be denoted by ωt, and the displacement, Op, of the point p

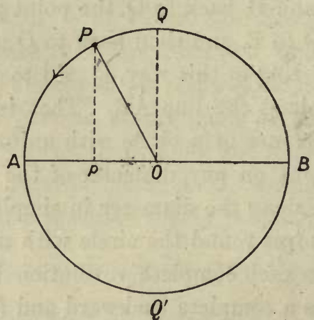


Fig. 36.

by OA sin ωt. For in the figure we have $\frac{Op}{OP} = \sin OPp$ or $Op = OP \sin OPp$. But OP $=$ OA, and OPp $=$ POQ $= \omega t$, and, therefore, Op $=$ OA sin ωt.

Now, if r denote the amplitude and t the period of the motion, we have OA $= r$ and $T = \frac{2\pi}{\omega}$ or $\omega = \frac{2\pi}{T}$. It follows, therefore, that

$$Op = r \sin \frac{2\pi t}{T}.$$

This result shows that during a complete period—that is, as t changes from 0 to T—the displacement OP varies in the same way as the sine of an angle varies as the magnitude of the angle changes from 0 to 2π. This law of the variation of the displacement with time during each complete period is *the* characteristic of simple harmonic motion.

If we plot a curve showing how the value of $r \sin \frac{2\pi t}{T}$ varies with t, the ordinate of the curve gives the displacement, Op, of the point p at any instant during the motion, and the curve, shown in Fig. 37, is called the *displacement curve.* It is readily plotted in any particular case by finding graphically, as in Fig. 36, the values of Op for a number of successive positions of the point P, and then plotting these values as ordinates and the corresponding

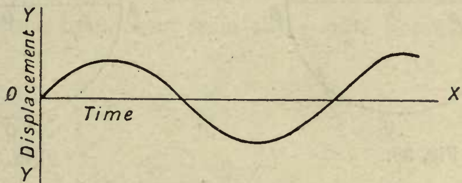


Fig. 37.

values of t as abscissæ. The curve is similar in form to the curve showing how the sine of an angle varies with the angle, and is sometimes called the *sine curve* or *the curve of sines.*

The velocity of the point p at any instant is evidently the component of the velocity of P in a direction parallel to AB. Thus, in Fig. 38, if the velocity of P is represented by PT the velocity of p will be represented by PR. That is, if the velocity of P is denoted by v the velocity of p is denoted by v cos TPR or $v \cos \omega t$, for TPR = SPO = POQ = ωt. This velocity may, like the displacement, be expressed in terms of r, the amplitude, and T, the period of the motion, for we have $v = \frac{2\pi r}{T}$, and $\omega = \frac{2\pi}{T}$, so that $v \cos \omega t$ may be written as $\frac{2\pi r}{T} \cos \frac{2\pi t}{T}$. It will be seen from this result that the velocity varies as the *cosine* of the angle $\frac{2\pi t}{T}$, while the displacement varies as the *sine* of the same angle. That is, the velocity has its maximum value $\frac{2\pi r}{T}$ when the displacement is zero, and the displacement has its maximum value r when the velocity is zero.

It should be noticed that the maximum velocity of p when at O, the middle point of its path, is the same as the velocity of the

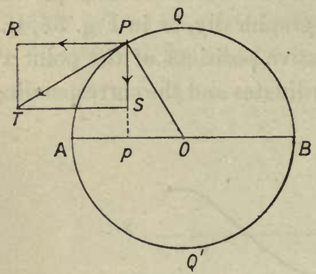


Fig. 38.

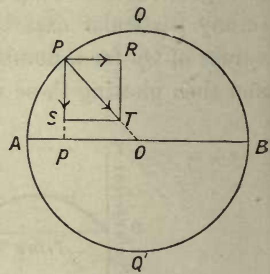


Fig. 39.

point P round the circle, for when P is at Q or Q′, the direction of its velocity is parallel to AB.

In the same way the acceleration of the point p is the component of the acceleration of P in a direction parallel to AB. Since P moves in a circle with uniform speed v it is subject to an acceleration of constant magnitude $\frac{v^2}{r}$ directed towards the centre of the circle. Thus, in Fig 39, if the acceleration of P is represented by PT, the acceleration of p is represented by PR. That is, the acceleration of p is always directed towards O, the centre of its path of motion, and is equal to $\frac{v^2}{r}\sin\omega t$ or $\frac{4\pi^2 r}{T^2}\sin\frac{2\pi t}{T}$.

The acceleration of the point p thus varies in the same way as the displacement, and is, in fact, directly proportional to the displacement, for $\frac{4\pi^2 r}{T^2} \sin \frac{2\pi t}{T} = \frac{4\pi^2}{T^2}\left(r \sin \frac{2\pi t}{T}\right)$, and $r \sin \frac{2\pi t}{T}$ denotes the displacement of the point. That is, if x denote the displacement at any instant of a point moving in simple harmonic motion, then the acceleration of the point at that instant is $\left(\frac{2\pi}{T}\right)^2 x$, where T denotes the period of the motion.

Hence, if a point move in simple harmonic motion of period T and amplitude r, the displacement, velocity, and acceleration of the point at the end of any time t, reckoned from an instant when the point passes through the centre of its path in the positive direction, have the values given below.

Let the displacement be denoted by x, the velocity by u, and the acceleration by a, then, from the results obtained above we have

$$x = r \sin \frac{2\pi t}{T}.$$

$$u = \frac{2\pi r}{T} \cos \frac{2\pi t}{T}.$$

$$a = \frac{4\pi^2 r}{T^2} \sin \frac{2\pi t}{T} = \left(\frac{2\pi}{T}\right)^2 x.$$

The relation $a = \left(\frac{2\pi}{T}\right)^2 x$ is an important one. It indicates that the acceleration a is directly proportional to the displacement x, and it can be seen from Fig. 39 that it is always directed towards the centre of the path of motion.

34. **The Simple Pendulum.**—A simple pendulum consists of a particle suspended by a thread so fine that its mass and weight may be neglected.

The particle forms the *bob* of the pendulum, and the length of the suspension thread from the point of suspension to the particle is called the *length* of the pendulum.

When the pendulum is set in vibration the particle which forms the *bob* of the pendulum swings backwards and forwards through an arc of a circle whose centre is at the point of suspension. If this arc is *small* compared with the length of the pendulum, the motion of the pendulum is practically isochronous, and approximates very closely to simple harmonic motion.

The period of vibration of a simple pendulum may be determined theoretically by the following method.

Let OP, Fig. 40, represent a simple pendulum suspended from the point O, and let the particle P oscillate backwards and

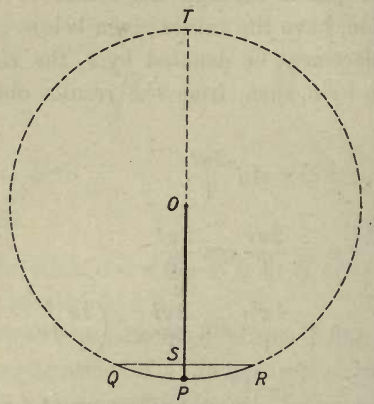


Fig. 40.

forwards through the small arc QPR. If this arc is very small the motion of the particle may be considered without sensible error as simple harmonic motion along QSR, the chord of the arc.

The velocity of the particle at P, the middle point of its path, is, therefore (Art. 33), the same as the velocity of a particle which moves round a circle described on QR as diameter with uniform speed, and makes a complete revolution in the time of one complete vibration of the particle. Hence, if SR, the radius of this circle, be denoted by r, the velocity of the particle at the middle point of its path by v, and the period of vibration of the pendulum by t, we have $v = \frac{2\pi r}{t}$.

But the velocity of the particle at P, the middle point of its path, is the velocity acquired in falling from R to P through the vertical distance SP. Hence, if SP be denoted by h, and the acceleration due to gravity by g, we have, by Art. 29, the relation,

$$v^2 = 2gh.$$

That is,

$$\left(\frac{2\pi r}{t^2}\right)^2 = 2gh,$$

or

$$\frac{4\pi^2 r^2}{t^2} = 2gh.$$

Now, in Fig. 40 we have TS . SP = $(\text{SR})^2$ by *Euc.* iii. 35. Hence, if OP, the length of the pendulum, be denoted by l, we have

$$(2l - h)\,h = r^2,$$

or

$$2lh - h^2 = r^2.$$

But when the arc QPR is very small, h is very small, and h^2 may be neglected. That is, when the amplitude of vibration of the pendulum is very small, we have

$$r^2 = 2lh;$$

and if we substitute this value of r^2 in the relation obtained above, we have

$$\frac{8\pi^2 lh}{t^2} = 2gh,$$

or

$$\frac{4\pi^2 l}{t^2} = g.$$

That is,

$$t^2 = \frac{4\pi^2 l}{g},$$

or

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

This result shows that for vibrations of very small amplitude the period of vibration of a simple pendulum is constant for a given length, and varies directly as the square root of the length for different lengths. For example, if the length of one simple pendulum is four times the length of another, the period of vibration of the longer pendulum will be twice that of the shorter.

Experiment 3.—Set up a simple pendulum and find its period of vibration when the length of the pendulum has, in turn, the following values:—100 cms., 81 cms., 64 cms., 49 cms., 36 cms., and 25 cms.

The period of vibration of the pendulum is most conveniently observed by noting the average time occupied by 10, 20, 50, or 100 complete vibrations. The vibrations should be counted, beginning at 0, and proceeding 0, 1, 2, 3, &c., as the bob of the pendulum passes in a given direction (say to the right) through the middle point of its swing.

It will be found that the periods of vibration thus found are in the ratio $\sqrt{100} : \sqrt{81} : \sqrt{64} : \sqrt{49} : \sqrt{36} : \sqrt{25}$, or 10 : 9 : 8 : 7 : 6 : 5. That is, the period of vibration is found to be directly proportional to the square root of the length when the length varies.

It will be seen that if t, the period of vibration of a simple pendulum of known length, l, is determined accurately by experiment, the value of g, the acceleration due to gravity, can be at once calculated from the relation

$$\frac{4\,\pi^2 l}{t^2} = g.$$

It must be remembered, however, that a simple pendulum is a theoretical conception and cannot be realised in practice. The nearest approach to it for practical purposes is a small heavy sphere, such as a single shot, or small bullet, suspended by a very fine thread or fibre. A rough determination of the acceleration due to gravity may be made by a simple pendulum of this kind. The length of the pendulum, from the point of suspension to the centre of the bob, is carefully measured as the pendulum hangs ready for use, and the period of vibration is determined as accurately as possible by the method explained above. The value of g can then be calculated from the relation given above.

The most accurate experimental determinations of the acceleration due to gravity are made by means of the **compound pendulum.** Any rigid body mounted so as to be capable of vibration round a fixed axis under the action of its weight is called a compound pendulum. If the body is of regular form, such that its moment of inertia (Art. 45), can be calculated from its dimensions, the length of the *equivalent simple pendulum*, which has the same period of vibration as the body, can be calculated from the radius of gyration of the body about an axis through the centre of gravity and the distance of the centre of gravity of the body from the axis of rotation.

Hence, if the period of vibration of a compound pendulum be determined with great accuracy by special methods, and the length of the equivalent simple pendulum is calculated from data found by exact measurement of the necessary dimensions of the pendulum, the acceleration due to gravity can be found by the relation given above.

The theory of the compound pendulum and the details of the methods of determining the acceleration due to gravity by its use are, however, beyond the scope of this book, and cannot be further considered.

Example.—Find the length of the seconds pendulum at a place where the acceleration due to gravity is 32·18 ft.-per-sec. per second.

The "seconds" pendulum is the simple pendulum which would *beat* seconds or make *half* a complete vibration in one second. The period of vibration of the seconds pendulum is, therefore, 2 seconds, so that by applying the relation

$$\frac{4\pi^2 l}{t^2} = g,$$

or

$$l = \frac{gt^2}{4\pi^2},$$

we have
$$l = \frac{32 \cdot 18 \times 4}{4 \times 9 \cdot 8696},$$

or
$$l = 3 \cdot 2604.$$

That is, the required length of the seconds pendulum at a place when the accumulation due to gravity is 32·18 ft.-per-sec. per second is 3·2604 feet, or nearly 39·125 inches.

The length of the seconds pendulum at London is 39·129 inches.

CHAPTER IX.

FORCE.

35. **Newton's First Law of Motion.**—Newton's first law of motion states that **a body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by the action of force to change this state.**

It should be remembered in interpreting this law that "rest" and "uniform motion in a straight line" are the two states in which there is *no change* in the motion of the body; that is, no change in the magnitude or direction of the velocity of the body. A body at rest is obviously free from change in this respect, and a body in uniform motion in a straight line moves with a velocity which is constant in magnitude and direction, and is, therefore, also free from change. Hence if a body is not acted on by "force" it must be in one of these states, and must continue, without change, in that state.

If the law is considered in relation to the body in motion it may be taken as defining a general property of matter. It states that any body or piece of matter is unable by its own action to change its state of rest or motion; unless acted on by an external "force" it remains at rest or continues to move with uniform motion in a straight line. This property of matter is called **inertia.**

If the law is considered in relation to the motion of the body it serves to define "force." The law states that the action of force compels a body to change its state of rest or motion. Force may, therefore, be defined as that which causes *change* in the motion of a body. It is important to understand that the change of motion produced by the action of a force is not a sudden change produced when the force first acts on the body, but a continuous progressive change which goes on during the whole time the force acts.

This law is sometimes used as a means of defining what is meant by *equal* intervals of time. It will be seen that *equal times* may be defined as the times in which a body free from the action of force moves over *equal distances.*

The law cannot be verified directly by experiment, for it is impossible in practice to free a body entirely from the action of force. General observation shows, however, that the more completely a body is freed from the action of force the less appreciable becomes the change in its state of motion.

For example, in the case of the motion of a small truck on straight horizontal rails, it is found that the more the opposing forces of friction are reduced the more nearly does the motion of the truck approximate to uniform motion in a straight line.

We are familiar from everyday experience with the fact that a body in motion tends to continue in motion till stopped by the application of force. If a train or carriage of any kind in rapid motion is stopped suddenly,* the passengers and luggage in the carriage tend to continue their onward motion. If they are securely fixed in position they are brought to rest with the train by the resistance of their supports; if they are not securely fixed they may actually continue their motion after the train has stopped until they are brought to rest by the resistance of some fixed support. That is, they are apparently thrown forward against whatever may be in front of them.

* It may be noticed that when a train is stopped gradually in the ordinary way, the passengers frequently experience a jerk *backwards* instead of forwards. This is due to the fact that as the train slows up the muscles of the body are braced to reduce the forward motion of the parts which are not directly supported, and the action of these muscles produces the jerk backwards if the train stops before they can be relaxed.

When a ball is thrown vertically upwards inside a railway carriage in motion it is, while in the air, practically free from the action of any force affecting its motion in a *horizontal* direction. Its horizontal motion, which is the same as that of the train at the instant of leaving the hand, must, therefore, continue *without change* while the ball is in the air. That is, if the motion of the train does not change while the ball is in the air, the two move forward together with the same horizontal motion. We know from experience that this is the case; if the ball is thrown vertically upwards from the hand, it keeps vertically over the hand while in the air, and ultimately returns to the hand.

It is also a matter of common observation that in order to set a body at rest in motion, the force applied must be greater than that required to overcome the frictional and other forces opposing the motion. The excess of force is required to produce the change from rest to motion, and if the excess is maintained the change of motion continues, and the velocity of the body steadily increases. If the excess is not maintained, and the applied force is reduced to equality with the opposing forces when a certain velocity has been attained, the motion continues without further change, for the resultant of the forces acting on the body is zero. If the applied force is removed, or reduced so as to be less than the opposing forces,* the body is gradually brought to rest again.

36. **Newton's Second Law of Motion.**—Newton's second law of motion states that **change of motion is proportional to the impressed force, and takes place in the direction in which that force acts.**

It is evident that the term "motion," as used in this law, applies to a *measurable quantity*, for it is said to be *proportional* to the impressed force. Newton explains that the term "motion,"

* These forces are here supposed to act only while the body is in motion.

as here used, involves the mass as well as the velocity of the body in motion, and must be taken, when applied to any body, as the product of the mass of the body into its velocity at the instant considered. That is, if m denote the mass of any body, and v its velocity at any instant, then the quantity mv is the *motion* of the body at that instant in the sense in which Newton uses the term in his statement of this second law. This quantity is now generally called **momentum.** It is a vector quantity, and its direction at any instant is the same as the direction of the velocity at that instant.

It will be seen, too, that the element of time must also be involved in the law, for the "change of motion" produced by the action of the impressed force, must depend upon the time in which the change is supposed to take place. In accordance with Newton's explanations on this point we may interpret "change of motion" to mean change of motion per unit time, or the time-rate of change of motion. We may now re-state the law in the following terms.

The time-rate of change of momentum of a body at any instant is directly proportional to the force acting on the body at that instant, and takes place in the direction in which the force acts.

If the force acting on the body is constant in magnitude and direction, the rate of change of momentum is also constant in magnitude and direction, but if the force is variable the rate of change of momentum also varies from instant to instant.

If at any instant *the rate of change of velocity*, or the *acceleration* of a body of mass m is denoted by a, the rate of change of momentum of the body must be ma; that is, the force acting on the body at that instant is directly proportional to ma, and the direction of the force is the same as the direction of the acceleration.

The simplest case illustrative of this general result is that in which a body moves along a straight line under the action of a constant force acting in the same direction as that in which the body is moving. In this case let u denote the velocity of the body at any instant, and v velocity at an instant t units of time later; then, if m denote the mass of the body the change of momentum during the time is given by $(mv - mu)$, and the rate of change of momentum by $\frac{mv - mu}{t}$, or $\frac{m(v-u)}{t}$. The force acting on the body is, therefore, proportional to $\frac{m(v-u)}{t}$, and since the force is constant, it follows that $\frac{v-u}{t}$ is constant. But $\frac{v-u}{t}$ is evidently the average acceleration of the body for the time t, and if this is constant for all values of t, it follows that the body moves with uniform acceleration along the straight line. Hence, if a body of mass m moves with uniform acceleration a along a straight line, the force acting on it is constant, and proportional to ma.

The motion of a body falling vertically under the action of its own weight is an example of this case of motion. The weight of the body is practically constant for a short fall, and the body is known by experiment to fall vertically in a straight line with the uniform acceleration known as the acceleration due to gravity.

In connection with the general result that the force acting on a body at any instant is directly proportional to the product of the mass of the body into the acceleration at that instant, it must be remembered that the direction of the force is, in all cases, the same as the direction of the acceleration.

37. **Unit Force.**—In the foregoing article it has been shown that if a body of mass m is subject at any instant to an acceleration a, the force acting on the body at that instant is proportional to ma, in accordance with Newton's second law of

motion. We may, therefore, write

$$F \text{ is proportional to } ma,$$

or

$$F = kma,$$

where k is a constant.

If we now agree that when m and a are both of unit value, F *shall also be of unit value*, the value of k becomes equal to 1, and we may write

$$F = ma.$$

That is, if we define **the unit of force as that force which produces unit acceleration in unit mass**, we may measure the force which produces an acceleration of a units in a mass of m units by the formula,

$$F = ma$$

as given above.

The unit of force in the English F.P.S. system of units will be that force which produces an acceleration of 1 ft.-per-sec. per sec. in a mass of 1 pound. This unit is called a **poundal**.

The unit of force in the C.G.S. system will, similarly, be that force which produces an acceleration of 1 cm.-per-sec. per second in a mass of 1 gramme. This unit is called the **dyne**.

It should be noticed that the relation $F = kma$, derived from Newton's second law, takes the from $F = ma$ as the result of the definition adopted for the unit of force.

It follows, therefore, in using the formula $F = ma$, that F will always be expressed in terms of a unit of force consistent with the units in which m and a are expressed. Thus, if m is in pounds, and a in feet-per-sec. per sec., F will be in poundals. Similarly, if m is in grammes, and a in cms.-per-sec. per sec., F will be in dynes.

38. **Mass.**—The meaning of the term mass can now be more fully explained and understood. It must be remembered that forces can be specified and compared without involving in any way the idea of mass. A force may, for example, be definitely

specified as the force which will extend a certain standard spiral spring through a given distance.

Similarly, equal forces may be defined as forces which extend the same spiral spring to the same extent, and unequal forces may be compared by comparing the extents to which they extend the same spring.

The mass of a body is evidently the quantity which measures the inertia of the body. In explaining Newton's second law of motion the term mass has been used without any explanation. It will be seen, however, that the idea of mass as a measurable quantity is derived from this law, and that the interpretation of the law includes the explanation of what is really meant by mass.

From the relation $F = kma$ deduced from the second law in Art. 37 above, we get

$$m = \frac{1}{k} \quad \frac{F}{a}.$$

This result shows that the mass of a body, as the term is used in this law, is a quantity which is directly proportional to the force required to produce a given acceleration of the body, or inversely proportional to the acceleration produced by a given force. The three statements given below follow directly from this result.

1. A mass may be definitely specified as the mass on which the action of a given force produces a given acceleration.

2. Equal masses may be defined as masses on which the action of the same force produces the same acceleration.

3. Masses may be compared (*a*) by comparing the accelerations produced by the action of the *same force* on the masses to be compared; or (*b*) by comparing the forces which, when acting on the masses, produce the *same acceleration*. In the one case (*a*) the masses would be inversely proportional to the accelerations, and in the other case (*b*), the masses would be directly proportional to the forces.

It will be seen at once from the last of these three statements [3 (*b*)] that masses may be compared by comparing their weights, **provided the acceleration due to gravity is the same for all** bodies whatever may be the size or material of the bodies. If this is the case the weights of different masses are evidently forces which produce the *same acceleration* when acting on the masses, and are, therefore, directly proportional to the masses.

Galileo and Newton both proved *by direct experiment* that the acceleration due to gravity is the same for all bodies, and is quite independent of the size and material of a body. It follows that the comparison of masses by the process of "weighing," as described in Art. 23, is in strict accordance with the definition of mass derived from Newton's second law.

It should be noticed that we can use the relation, $F = kma$, which we derived from this second law to define a unit of force, as above, or to define a unit of mass. If we selected a unit of length, a unit of time, and *a unit of force* (such as the force required to extend a standard spiral spring a specified distance) as fundamental units, we might define the unit of mass as the mass in which the unit of force produces unit acceleration. If, on the other hand, we follow the general practice and select a unit of length, a unit of time, and a *unit of mass* (Art. 4) as fundamental units, we define unit force as that force which produces unit acceleration by its action on unit mass, as explained above.

In either case the relation $F = kma$ would reduce to $F = ma$, but the system of units would be essentially different in the two cases.

39. **Weight.**—It has already been explained that the weight of a body at any place on the earth's surface is the *force* with which the earth attracts the body towards it. This force acts towards the centre of the earth and its direction at any place determines the *vertical* direction at that place.

When a body falls at any place the force acting on it is its own weight and the acceleration with which it falls is the acceleration due to gravity at the place.

If, therefore, we apply the formula $F = ma$ to the case of a body falling freely at any place, and if we write W instead of F for the particular force called weight, we get the formula

$$W = mg,$$

where g denotes the acceleration due to gravity at the place. In this formula W evidently denotes the weight of the body at the given place in absolute units of force. The weight of a pound mass at a place where the acceleration due to gravity is 32·18 ft.-per-sec. per sec. is evidently $(1 \times 32{\cdot}18)$ or **32·18 poundals.** It will be seen from this that a poundal is, roughly, equal to the *weight* of half an ounce.

Similarly, the weight of a gramme at a place where the acceleration due to gravity is 980·8 cms.-per-sec. per sec. is **980·8 dynes.**

It will be clear from what has been said that the weight of a given mass depends for its value at any place on the acceleration due to gravity at that place, and varies, therefore, from place to place on the earth's surface.

The weight of a pound or **a pound-weight** is, therefore, not a constant force but is equal at any place to g **poundals**, where g is the acceleration due to gravity in ft.-per-sec. per sec. at that place.

Similarly, the weight of a gramme or a **gramme-weight** is not a constant force, but is equal at any place to g **dynes**, where g is the acceleration due to gravity in cms.-per-sec. per sec. at that place.

When a force is expressed in terms of the *weight* of some convenient unit of mass, such as a *gramme-weight*, a *pound-weight*, a *ton-weight*, or some similar unit, it is said to be expressed in **gravitation units.** These units are very generally employed in engineering problems, but it must be remembered in dealing with them that the value of a gravitation unit is not a constant, but depends for its value at any place on the acceleration due to gravity at that place.

The relation $W = mg$ may be used conveniently to show that the weights of different bodies at the same place are directly proportional to their masses. Let m_1 and m_2 denote the masses of two bodies, and let W_1 and W_2 denote the weights of these masses at a given place. Then, if g denote the acceleration due to gravity at the place, we have

$$W_1 = m_1 g$$

and

$$W_2 = m_2 g.$$

This gives

$$\frac{W_1}{W_2} = \frac{m_1 g}{m_2 g},$$

and since g is known, from experimental evidence, to have *the same value for both masses* this relation reduces to

$$\frac{W_1}{W_2} = \frac{m_1}{m_2}.$$

That is, the weights of the two bodies are directly proportional to their masses.

40. **Atwood's Machine.**—Atwood's machine is a piece of apparatus designed for the experimental study of the relations between force, mass, and acceleration, which are involved in Newton's second law of motion.

It consists, essentially, of a light wheel mounted with its axis horizontal, on bearings which are constructed so as to be as free from friction as it is possible to make them. The rim of the wheel is grooved, like a pulley, so that a light flexible cord carrying a mass at each end, can be passed over it, as shown in Fig. 41. If the two masses, A and B, are equal, their weights are equal, and will evidently balance each other, so that the system remains at rest. If, however, we place a small mass as a rider on one of these equal masses, the weight of this small mass will set the system made up of the two masses A and B, the rider, the cord joining the masses, and the wheel, in motion. If, however, we neglect the motion of the cord and the wheel, and neglect also the forces of friction and air resistance which oppose motion, we may consider that the weight of the rider acts on a mass equal to the sum of the two masses, A and B and the mass of the rider, and sets it in motion. Hence, if m denote the mass of the rider, and M the mass of each of the two equal masses A and B, we have a force of mg units acting on a mass of $(2M + m)$ units, and we may, therefore, apply the relation $F = ma$, and write

$$mg = (2M + m)\,a,$$

where a denotes the acceleration of the mass in motion.

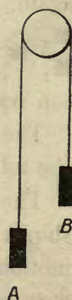


Fig. 41.

If we now measure the acceleration a, *experimentally*, for a number of different values of m and M, we can examine whether the values obtained by experiment are consistent with the relation given above. For example, in one set of experiments let m be varied, but let the mass moved $(2M + m)$ be kept constant and equal to **M**. Then if m_1 and m_2 denote the masses of the rider, in two cases the relation deduced from Newton's second law gives

$$m_1 g = \mathbf{M} a_1,$$

and

$$m_2 g = \mathbf{M} a_2,$$

or,

$$\frac{m_1}{m_2} = \frac{a_1}{a_2},$$

where a_1 and a_2 are the accelerations in the two cases. If, now, the values of a_1 and a_2 are determined by experiment, and the value of the ratio $\frac{m_1}{m_2}$ is known,* we can readily test whether the experimental results are in accord with the result obtained

* The ratios $\frac{m_1}{m_2}$ and $\frac{M_2}{M_1}$ cannot, in these experiments, be determined

above. Similarly, in another set of experiments let m be kept constant, but let the mass moved $(2\mathbf{M}+m)$ be varied. Then, if $\mathbf{M}_1$ and $\mathbf{M}_2$ denote the value of the mass moved, in two cases we have—

$$mg = \mathbf{M}_1 a_1,$$

and

$$mg = \mathbf{M}_2 a_2,$$

or,

$$\frac{a_1}{a_2} = \frac{\mathbf{M}_2}{\mathbf{M}_1},$$

where a_1 and a_2 are the accelerations in the two cases. Here again, if a_1 and a_2 are found by experiment, and the ratio $\frac{\mathbf{M}_2}{\mathbf{M}_1}$ is known,* the agreement between theory and experiment can readily be tested.

The acceleration of the moving mass may be determined by the following method.

The masses A and B, with the rider on A, are arranged as shown in Fig. 42, and a vertical scale, SS, is set up so that the motion of either mass may be followed on it. The system is first held at rest by supporting the mass A on a small movable platform. At a marked instant this platform is removed, and the system is allowed to move for a definite time under the action of the weight of the rider as the moving force. At the end of this time the cylindrical mass A passes through the ring

by the process of weighing without assuming the truth of the result it is desired to test. The masses used in the experiment may, however, be made up of small equal masses made of the same volume of the same material. If some of these small equal masses are made in the form of riders, and others as slotted discs and carriers, or in some other suitable form, the experiments can be conveniently carried out by simply dividing the masses between A and B (Fig. 41), so as to fulfil any given conditions, such as those indicated above, and the ratios $\frac{m_1}{m_2}$ and $\frac{\mathbf{M}_2}{\mathbf{M}_1}$ can be determined without making any assumption other than the permissible one, that the masses of equal volume of the same material are equal.

at C, when the rider is removed and left resting on the ring without interfering with the motion of the masses A and B. The instant at which the rider is lifted off A is marked or recorded in some convenient way.

After the removal of the rider, the system is allowed to continue in motion for another interval of time, until it is brought to rest by the mass A striking the small platform fixed at D. The instant at which A touches this platform is marked or recorded as before.

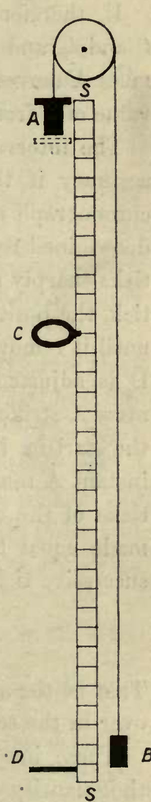


Fig. 42.

In this experiment three instants are recorded —the starting instant, the instant at which the rider is removed from A, and the instant at which A touches the platform at D. During the interval between the first and second of these instants, the masses move with uniform acceleration, under the action of the weight of the rider as the moving force. During the interval between the second and third of these instants the system is free from the action of any force tending to change its state of motion (friction and air resistance being neglected), and moves, therefore, throughout the time with a uniform velocity equal to the velocity acquired at the instant the rider was removed. Hence, if t_1 denotes the first of these intervals of time, and a denotes the acceleration of the moving mass during this interval, we have—

$$v = at_1,$$

where v denotes the velocity acquired at the instant the rider is removed. Then, if t_2 denotes the second interval of time in which the masses move with uniform velocity, v, the space passed over in the time, is given by

$$s = vt_2.$$

That is, $$s = at_1 \cdot t_2 = at_1t_2,$$

and $$a = \frac{s}{t_1\, t_2}.$$

If, therefore, in this experiment we determine the intervals t_1 and t_2, and observe on the scale SS the distance s which the mass A moves through during the time t_2, we can calculate the value of a from the relation given above.

The intervals of time t_1 and t_2 can be determined with fair accuracy if the instants referred to above are recorded on a chronograph trace as explained in Art. 19. They may also be determined roughly with the aid of a clock or metronome which ticks sharply and distinctly. The mass A is released at a certain tick, the position of the ring at C is adjusted by repeated trials until it removes the rider at the next tick, and the position of D is adjusted in the same way until it is so placed that the mass A strikes it at the instant the third tick is heard. That is, the starting instant, the instant of removing the rider, and the instant A touches D, are made to coincide with three successive ticks of the clock or metronome. The times t_1 and t_2 are thus made equal to one another, and to the interval between two successive ticks. If this interval is exactly one second we have

$$a = \frac{s}{1 \times 1}, \text{or } a = s.$$

That is, the acceleration is *numerically* equal to the space passed over in the second interval of time.

These intervals of time may also be measured by weighing the quantity of water or mercury which escapes in each interval through a small hole in a vessel containing the liquid. This is the *water clock* method adopted by early experimenters with Atwood's machine.

The distance s is obviously the distance between the upper faces of the ring at C and the platform at D, diminished by the length of mass A.

If the experiments indicated above are carried out in the manner here described, it will be found that the more completely the sources of error in the experiments are removed the more nearly do the results obtained agree with those deduced from Newton's second law of motion.

It will be seen that if we accept Newton's second law of motion as true, we can use Atwood's machine to determine the acceleration due to gravity. From the relation

$$mg = (2M + m)\,a,$$

we get

$$g = \frac{2M + m}{m} \cdot a;$$

so that if a is determined by experiment, as explained above, and M and m are known, the value of g can be calculated directly from this result.

Atwood's machine is of great interest historically and theoretically, but it is extremely difficult to obtain anything like accurate results by its use for experimental work. The main sources of error are those due to the neglect of (a) the mass of the string; (b) the inertia of the wheel; and (c) the friction and air resistance which oppose motion. The moving force (the weight of the rider) should really be split up into the following parts:—

f_1, which acts on the masses A and B and the mass of the rider, and which produces the acceleration of these masses.

x_1, which acts on the mass of the string, and sets it in motion with the acceleration of the system.

x_2, which acts on the wheel and sets it in rotation round its axis with an angular acceleration, such that the linear acceleration of a point on the rim where the string touches the wheel is the same as that of the string.

x_3, which is neutralised by the opposing forces of friction and air resistance.

Of these f_1 is the real moving force which produces the acceleration of the moving system, and it will be seen that $f_1 = f - (x_1 + x_2 + x_3)$.

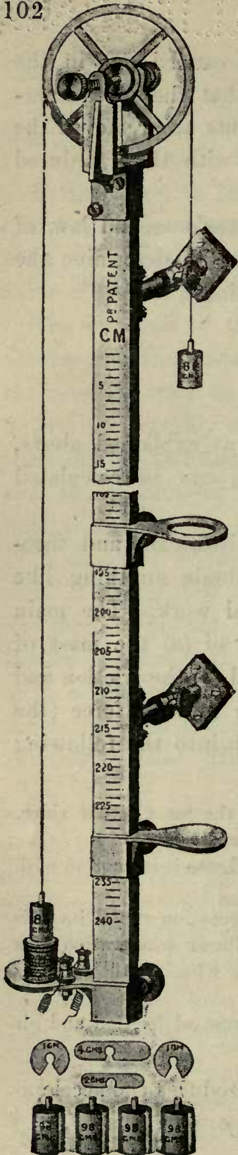


Fig. 43.

The error which results from neglecting x_1, x_2, and x_3, and taking f, the weight of the rider, as the moving force, will depend upon the magnitude of $(x_1 + x_2 + x_3)$, compared with f, and may be very great. If the string is very light, and the wheel is light, properly designed, and mounted on frictionless bearings, x_1, x_2, and x_3, may, for rough purposes, be neglected, provided f is not small. For more accurate work, however, these errors must be corrected and allowed for by methods which we cannot here consider. Correction should also be made for the change in the moving force caused by the fact that as the system moves the weight of the string on one side of the wheel increases, while the weight on the other side decreases.

In all elementary problems and experiments relating to Atwood's machine, the mass of the string, the inertia of the wheel, and the opposing forces of friction, are usually neglected.

A form of Atwood's machine suitable for accurate work is shown in Fig. 43. The experimental results which can be obtained by this machine, even under the best conditions and with due regard to the sources of error, are not, however, of a very high degree of accuracy.

41. **Newton's Third Law of Motion.** — Newton's third law of

motion states that **to every action there is an equal and opposite reaction; or that the mutual actions of two bodies are equal in magnitude and opposite in direction.**

The terms "action" and "reaction" in this law apply to forces. The law implies that force can be exerted only by one piece of matter on another, and that the action between any two bodies is mutual, so that each body may be considered to exert force on the other. This mutual action between two bodies is generally called *stress.* Hence, if a stress exists between two bodies A and B, and the force exerted by A on B is taken as the *action,* then the equal and opposite force exerted by B on A is called the *reaction.* Action and reaction are thus merely opposite aspects of the stress between the bodies.

When the stress between any two bodies is such that each presses against the other, the stress is of the particular kind known as *pressure.* Thus, if we press with the hand against a wall, the hand presses on the wall, and the wall resists or presses back against the hand. The stress between the hand and the wall is thus a pressure; the hand exerts pressure on the wall, and the wall exerts an equal and opposite pressure on the hand. That is, the action and reaction are equal and opposite.

Similarly, when a book rests on a table it exerts a pressure vertically downwards on the table; at the same time the table resists and exerts pressure vertically upwards on the book. This upward pressure exerted by the table on the book must be equal and opposite to the weight of the book, for the two forces *acting on the book* are its weight acting downwards, and the resistance or upward pressure of the table acting upwards, and since the book remains at rest, these forces must be exactly equal and opposite to each other. Hence, if the downward pressure of the book on the table be taken as the action, the resistance or upward pressure of the table on the book is the reaction, and these forces are equal and opposite in accordance with Newton's third law, and each is equal to the weight of the book.

If a box rests on the floor of a lift *in motion* the stress between the under surface of the box and the floor of the lift is a pressure, and the action and reaction between the surfaces are equal and opposite; but they are not necessarily equal to the weight of the box. Let the pressure of the box on the floor of the lift be denoted by P, then the reaction of the floor acts vertically upwards and is also equal to P, as shown in Fig. 44. The forces *acting on the box* are, therefore, its weight, W, acting vertically downwards, and P the resistance from the floor acting vertically upwards. Hence, if P = W, the resultant force acting on the box is of zero value, and the box (and lift) must either be at rest or moving with uniform motion in a straight line up or down. In this case the pressure of the box on the floor of the lift would be equal to its weight. If, however, W is greater than P, the box is under the action of a force (W — P) acting downwards, and will, therefore, be subject to the downward acceleration caused by the action of this force on the mass of the box. In this case the pressure of the box on the floor of the lift is *less* than its weight. The weight may, in fact, be divided into two parts—a part, P, which exerts pressure on the floor, and the remainder (W — P), which gives the box its *downward* acceleration. If, again, P is greater than W, the box is moving under the action of a force (P — W) acting upwards, and will be subject to the acceleration caused by the action of this force on the mass of the box. In this case the pressure of the box on the floor of the lift is *greater* than its weight, for the floor not only supports the weight of the box, but exerts the additional force (P — W) which gives the box its *upward* acceleration.

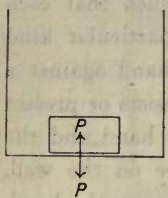


Fig. 44.

When the stress between any two bodies is such that each body exerts a pull towards itself on the other, the stress is of the type called *tension*. Thus, if two persons pull against each other along a rope, as in a "tug of war," the stress between them is a tension. The stress is properly considered to act across any transverse section of the rope between the two portions of the system separated by this section. If the rope is at rest the tension is practically the same at all points in its length. For, if T_1 denotes the tension at a point, A (Fig. 45), and T_2 the tension at another point, B, it is evident from the figure that the force *acting on the portion* AB of the rope * is $(T_1 - T_2)$, if we assume T_1 to be greater than T_2, and if we neglect the weight of the rope. But if the rope is at rest, the force acting on any portion of it must be of zero value—that is, T_1 and T_2 must be equal. Hence, when the rope is at rest, and

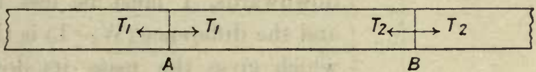


Fig. 45.

the tension the same at all points in it, we generally speak of the *tension of the rope*, and consider the stress to act by means of the rope between the two persons pulling at its ends. When a body hangs at rest from a nail by a thread, as in Fig. 46, the tension in the thread at any point, A, is equal to the weight of the body and the piece of thread below this point. For this weight, W, acting at A, is evidently balanced, as shown in the figure, by T, the tension in the string at this point. If we neglect the weight of the thread the tension is evidently the same at all points in the thread, and is equal to the weight of the body. The body thus exerts a downward pull, equal to its weight, on the nail, and the nail exerts an equal upward pull on the body. The thread in this case may be looked upon as the

* The left-hand arrow at A and the right-hand arrow at B evidently indicate the forces *acting on the portion* AB.

medium or connection between the body and the nail, by means of which the stress between these two bodies is maintained.

If a mass hanging by a thread is *in motion*, as in the case of the masses of Atwood's machine, the tension of the thread is not in general equal to the weight of the mass. Let the mass, A, in Fig. 47, be supposed to be moving up or down in a vertical line. The forces *acting on the mass* are its weight, W, acting vertically downwards, and the tension in the thread, T, acting vertically upwards. If the mass is at rest, as above, or moving up or down with *uniform velocity*, the resultant force acting on it must be of zero value, and T must be equal to W. That is, in this case, the tension in the string would be equal to the weight of the mass. If the mass is moving upwards or downwards with uniform acceleration downwards, T must be less than W and the difference (W — T) is the force which gives the mass its downward acceleration. If the mass is moving upwards or downwards with uniform acceleration upwards, T must be greater than W and the difference (T — W) is the force which gives the mass its upward acceleration.

Fig. 46.

Fig. 47.

It must be remembered that in each of these cases the stress at the point of attachment of the string to the mass is the tension in the string at that point; the string exerts an upward pull on the mass equal to the tension T, and the mass exerts an equal downward pull on the string. The weight W is an external force exerted by the earth on the mass; the reaction to this is the equal and opposite force exerted by the mass on the earth.

In applying the third law of motion to any body or system in motion it must be remembered that the action and reaction must be taken at the same point, or across the same section. Thus, if AB (Fig. 48) represents a body in motion in the direction of the arrow at B, and we consider the stress across a transverse section at C, we can say that the action and reaction at this section are equal and opposite. Similarly, if we consider the stress across the transverse section at D, we can also say that the action and reaction at this section are equal and opposite. The stress at D will not, however, in general be the same as at C. Let P and P′ denote the stresses at C and D respectively, then the resultant force *acting on the portion* CD is evidently the difference between P and P′.*

If the body is in motion with *uniform velocity*, P is equal to P′. If the body is subject to acceleration in the direction of its motion, then P must be greater than P′, and the difference

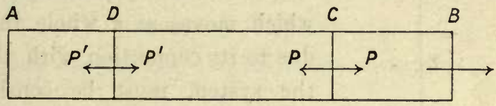


Fig. 48.

(P − P′) is the force which gives the mass of the portion CD the acceleration with which it moves. If the body is subject to *retardation* then P′ must be greater than P, and the difference (P′ − P) will be such as to produce in the mass of CD the retardation to which it is subject.

Thus, in the case of the motion of a horse and cart along a level road, the horse and cart move as one system, and the difference between the stresses at any two vertical sections of the system gives the force which at any instant determines the motion of the portion of the system between the sections. The force which causes the motion of the system as a whole is derived from the stress between the horse's feet and the

* External forces, such as the weight of the body and frictional resistance to motion, are here neglected.

ground. The component parallel to the road of the reaction of the ground on the horse's feet is the force which acts on the system at any instant, and the system moves forward with uniform velocity, or is subject to acceleration or retardation according as this force is equal to, or greater or less than, the forces which oppose motion.

The reaction of the ground on the horse's feet is equal and opposite to the "action" of the feet on the ground, and depends, therefore, for its value on the muscular effort exerted by the horse.

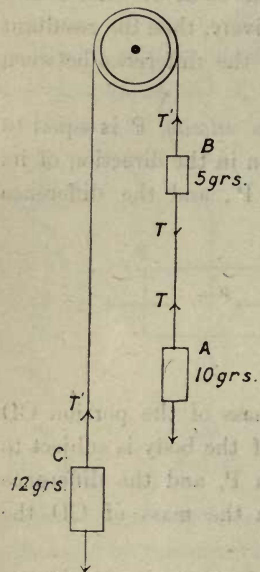


Fig. 49.

It will be understood from the examples which have been given above that, in determining the forces which act on any portion of a system which moves as a whole, the stresses due to its connection with the rest of the system must be considered, as well as the external forces which may act on it. This may be more fully understood from a study of the following numerical example.

Numerical Example.—Three masses, A, B, and C, of 10 grammes, 5 grammes, and 12 grammes mass respectively, are connected in line by a fine thread and arranged on the wheel of an Atwood's machine in the manner shown in Fig. 49; find the tension in the thread connecting A and B, and in the thread connecting B and C when the system is in motion under the action of the weights of the masses.

The thread being fine the mass and weight of the connecting threads may be neglected, and the tension in either thread may be considered to be the same at all points in its length.

Let T denote the tension in the thread connecting A and B, and T′ the tension in the thread connecting B and C. Consider first the motion of the mass A. The forces *acting on it* are its weight W, acting vertically downwards, and the tension T, in the thread AB, acting vertically upwards. The mass is evidently subject to downward acceleration, so that the resultant force acting on it is downwards and equal to (W − T). The mass of A is 10 grammes, and its weight, W, is (10 × 981) dynes (Art. 39), so that by applying the relation $F = ma$, we get

$$9{,}810 - T = 10a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where a is the acceleration of the mass A, and, therefore, of the whole system of masses.

Consider next the motion of the mass B. The forces *acting on this mass* are its weight, W′, acting vertically downwards, the tension, T, in the thread BA, also acting vertically downwards, and the tension, T′, in the thread BC, acting vertically upwards. The mass moves with downward acceleration, a, so that the resultant force acting on it is downward, and is equal to (W′ + T − T′). The mass of B is 5 grammes, and its weight W′, is (5 × 981) dynes, so that by again applying the relation $F = ma$, we get

$$4{,}905 + T - T' = 5a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the same way by considering the motion of the mass C, we get the result

$$T' - 11{,}772 = 12a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From the three equations thus obtained we can find the values of T, T′, and a in the usual way.

It will be found that

$$T = 8{,}720,$$
$$T' = 13{,}080,$$
and
$$a = 109.$$

T and T′ being expressed in dynes, and a in cms.-per-sec. per sec.

These results may be arrived at more expeditiously by considering first the system as a whole, and finding the acceleration of its motion.

The mass of the system is 27 grammes, and the force to which its acceleration is due, is evidently the difference between 15 gramme-weights and 12 gramme-weights, or 3 gramme-weights, or (3 × 981) dynes. If, therefore, we apply the relation $F = ma$ to the system, we get

$$3 \times 981 = 27a\,;$$
or
$$a = 109.$$

That is, the acceleration of the system is 109 cms.-per-sec. per sec.

If we substitute this value of a in equation (1) obtained above, we get

$$9{,}810 - T = 1{,}090;$$

or

$$T = 8{,}720.$$

That is, the tension in the thread AB is 8,720 dynes, or $8\frac{8}{9}$ gramme-weights. Similarly, by substituting for a in equation (3), we get

$$T' - 11{,}772 = 1{,}308;$$

or

$$T' = 13{,}080.$$

That is, the tension in the thread BC is 13,080 dynes, or $13\frac{1}{3}$ gramme-weights.

42. **Motion in a Circle.**—It has been shown in Art. 32 that a particle moving in a circle of radius r, with a velocity of constant magnitude v, is subject to a constant acceleration $\frac{v^2}{r}$ directed towards the centre of the circle. Hence, if m denote the mass of the particle, the magnitude of the force which acts on the particle and keeps it moving in its circular path is $\frac{mv^2}{r}$, and the direction of this force always passes through the centre of the circle.

For example, the moon moves round the earth in an approximately circular path, and the force constraining it to move in this path is the force of attraction exerted on it by the earth. Hence, if M denote the mass of the moon, V the magnitude of its velocity round the earth, and R its distance from the centre of the earth, the force of attraction exerted on it by the earth must be equal to $\frac{MV^2}{R}$.

Example.—A stone of 100 grammes mass is whirled round in a vertical circle, at the end of a string 100 cms. long, at a uniform rate of 10 complete revolutions per minute. Find the tension in the string when the stone is (a) at the top of its path, and (b) at the bottom of its path.

(a) Let W denote the weight of the stone, and T the tension in the string; then when the stone is at the top of its path *the force acting on the stone towards the centre of the circle* is evidently W + T. Hence, if m denote the mass of the stone, r the radius of the circle in which it moves, and v its velocity in this circular path, we must have

$$W + T = \frac{mv^2}{r}.$$

From the data of the question we know that in C.G.S. units we have

$$m = 100 \text{ (grammes).}$$
$$r = 100 \text{ (cms.)}$$
$$v = \frac{10 \times 200\pi}{60} = \frac{100\pi}{3} \text{ (cms. per sec.)}$$

and $$W = 100 \times 981 = 98{,}100 \text{ (dynes).}$$

Hence, we get,

$$98{,}100 + T = \frac{100 \times \pi^2 \times 10^4}{100} = \pi^2 \times 10^4.$$

That is,

$$\begin{aligned} T &= \pi^2 \times 10^4 - 98{,}100 \\ &= 98{,}696 - 98{,}100 \\ &= 596. \end{aligned}$$

Or the tension on the string when the stone is at the top of its path is 596 dynes, or ·61 gramme-weights.

(*b*) Similarly, when the stone is at the bottom of its path, the force *acting on the stone towards the centre of the circle* is T − W.

Hence, as above, we get

$$T - W = \frac{mv^2}{r}$$

or $$T - 98{,}100 = \pi^2 \times 10^4.$$

That is

$$\begin{aligned} T &= 98{,}100 + 98{,}696 \\ &= 196{,}796. \end{aligned}$$

That is, the tension on the string at the lowest point in its path is 196,796 dynes, or about 200·6 gramme-weights.

It is important to realise, in connection with the result obtained above, that $\frac{mv^2}{r}$ is not the magnitude of a new force which acts on the body in virtue of its circular motion, and in addition to any other forces which may be acting on it; it is the magnitude of the resultant of the forces actually acting on the body.

That is, if a body of mass m moves in a circle of radius r, with a velocity of constant magnitude v, the resultant force acting on the body at any point in its path is directed towards the centre of the circle, and its magnitude is $\frac{mv^2}{r}$.

When a particle moves in a circle, it may be said, in accordance with Newton's third law of motion, that a stress exists between the particle and the centre of the circle. This stress acts on the particle towards the centre, and on the centre towards the particle, or away from the centre. These two aspects of this stress have been called the *centripetal* and *centrifugal* forces.

The existence of this stress between a particle in circular motion and the centre of its path explains why a flywheel or emery-wheel in rapid rotation sometimes "bursts." Stress exists between every particle of the wheel and the axis of rotation, as a tension in the intervening material of the wheel, and if this stress becomes at any point too great for the strength of the material to withstand, the wheel "bursts" into fragments. At the instant of bursting each fragment is freed from the constraint which compels it to move in a circle, and will, therefore, continue its motion in the same direction and with the velocity which it has at that instant.

43. **Simple Harmonic Motion.**—It has been shown in Art. 33, that when a particle moves in simple harmonic motion, the acceleration of the particle is directed towards the centre of the path of motion, and is of magnitude $\left(\frac{2\pi}{T}\right)^2 x$, where T denotes the period of the motion and x the displacement of the particle. Hence, if m denote the mass of the particle, the force acting on the particle at any instant during its motion is $\left(\frac{2\pi}{T}\right)^2 mx$, or $\frac{4\pi^2 m}{T^2} x$.

That is, the force acting on the particle at any instant during its motion is directly proportional to its displacement at that instant, and is directed towards the centre of the path of motion.

Hence, if a body is so constrained that the force which acts on it, as the result of any displacement from its position of rest, is proportional to the displacement and directed towards its position of rest, the body will move in simple harmonic motion along the line of displacement, and its position of rest will be the centre of its path of motion. Also, if the force acting on the particle is denoted by kx, where k is a constant, we have

$$kx = \frac{4\pi^2 m}{T^2} . x,$$

or

$$k = \frac{4\pi^2 m}{T^2}.$$

For example, if a small bullet of mass m, hangs by a thin elastic cord which is stretched by the weight of the bullet through a distance, d, the force which will act on the bullet as the result of a small vertical displacement, x, from its position of rest will be* $\frac{mg}{d} . x$, and will be directed towards the position of rest. The bullet will, therefore, move up and down in simple harmonic motion, and as

$$\frac{mg}{d} = \frac{4\pi^2 m}{T^2},$$

the period of its motion is given by

$$T = 2\pi \sqrt{\frac{d}{g}}.$$

44. **Moment of a Force.**—The moment of a force about any point is defined as the product of the magnitude of the force into the length of the perpendicular from the point on to the

* The force causing *unit elongation* of the cord is denoted by $\frac{mg}{d}$; the force resulting from a change, x, in the elongation is, therefore, denoted by $\frac{mg}{d} . x$.

line of action of the force. Thus, in Fig. 50, if a force of magnitude, F, act along the line AB the moment of the force round the point O is measured by the product of F into the length of the perpendicular OP. That is, if the moment of the force be denoted by M, and the length of OP by d, we have $M = Fd$. The moment of the force F about the point O thus depends upon F, the magnitude of the force, and d, the length of the *arm* OP, and if either of these quantities is zero the moment is zero. When d is zero the point O is on the line of action of the force, so that the moment of a force round any point on its line of action is zero.

If the force F be supposed to act on a rigid body free to rotate round a fixed axis passing through O at right angles to the plane of the paper, the force tends to produce rotation of

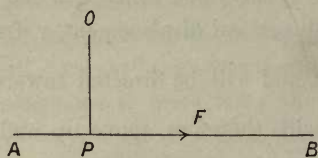


Fig. 50.

the body round this axis, and the moment of the force round O may be taken as a measure of the effect of the force in tending to produce rotation round the axis passing through O. In order to distinguish between the two possible directions of rotation a moment is considered to be of positive sign if it tends to produce rotation in a direction opposite to that of the hands of a clock, and of negative sign if it tends to produce rotation in the same direction as the hands of a clock.

It can be shown by experiment that two forces acting on the same body and tending to produce rotation in opposite directions round the same axis, will balance each other exactly if their moments round the axis of rotation are of equal magnitude. This shows that the moment of a force round a point is a real measure of the effect of the force in tending to produce rotation round the point. It follows also from this that if a number of forces * act on a body tending to produce rotation round the same axis, the total equivalent moment of the system of forces round the axis is the algebraic sum of the moments of the individual forces round the axis.

Experiment 4.—Take a flat strip of wood, such as a half-metre scale, and balance it on a knife-edge placed horizontally at right angles to the length of the scale. Now take masses of 100 grammes and 200 grammes and suspend them from the scale, one on each side of the knife-edge, by means of threads. The threads should be looped at their upper ends, so that the masses can be suspended by passing the loop over the scale. Adjust the positions of the masses on the scale until an exact balance is obtained and the scale balances on the knife-edge in a horizontal position. This is most conveniently done by setting the loop of the thread carrying the smaller mass at any convenient distance from the knife-edge and then sliding the loop of the thread carrying the other mass along the scale until a balance is obtained. It will then be found that the distances of the suspension loops from the knife-edge are inversely proportional to the masses carried by these loops. That is, if d_1 and d_2 denote the distances from the knife-edge of the points of suspension of the 100 grammes mass and the 200 grammes mass respectively, then $d_1 : d_2 :: 2 : 1$, and it will be found on trial that this relation is true for all corresponding values of d_1 and d_2. Similarly, it may be found by trial with other masses that the distances d_1 and d_2 are always inversely proportional to the masses. That is, m_1 and m_2 denote the masses, we always find that $d_1 : d_2 :: m_2 : m_1$. This experiment proves that moments tending to produce rotation round the same axis balance each other when they are of equal magnitude.

The scale is a rigid body free to rotate about the knife-edge as axis, and the weights of the suspended masses acting on it at the points of suspension tend to produce rotation round this axis in opposite directions. The axis is horizontal, and the weights act vertically, so that the weight of each mass acts in a plane at right angles to the axis. Hence, when the scale is horizontal, the moment of the weight

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* Each force is supposed to act in a plane at right angles to the axis, so that the forces considered are either all in the same plane or in parallel planes.

of each mass round the axis is the product of the magnitude of the weight into the distance of the point of suspension of the mass from the knife-edge. If, therefore, the masses are denoted by m_1 and m_2, and the distances of the points of suspension of these masses from the knife-edge by d_1 and d_2 respectively, the moments of the weights of the masses round the knife-edge are given by m_1gd_1 and m_2gd_2, where g denotes the acceleration due to gravity. Hence, if these moments are equal when a balance is obtained, we should have—

$$m_1gd_1 = m_2gd_2,$$

or

$$\frac{d_1}{d_2} = \frac{m_2}{m_1}.$$

This, however, is the result actually obtained by the experiment. It may, therefore, be considered as established that forces of equal moment round any axis have equal effects in tending to produce rotation round that axis.

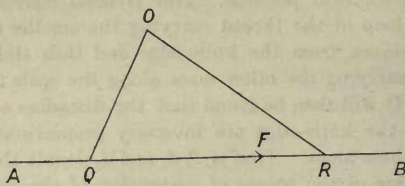

Fig. 51.

This principle, established by this experiment, is sometimes called the principle of moments.

If the force F in Fig. 50 is represented as in Fig. 51 by a length QR taken on its line of action, AB, the moment of the force round the point O will be represented by twice the area of the triangle QOR. For, by definition, the moment of the force round O is measured by the product of the measure of QR into the measure of OP, and this product is also the measure of twice the area of the triangle QOR.

It will be seen that, in dealing with forces whose lines of action all lie in the same plane, the moments of the forces round an axis at right angles to the plane become the moments of the forces round the point at which the plane cuts the axis.

45. **Motion of Rotation.**—If a body rotates round an axis with *angular acceleration* a, the linear acceleration of a particle of the body at a distance r from the axis is ra, and the force acting on the particle is mra, where m denotes the mass of the particle. The direction of the linear acceleration of the particle is tangential to the circle in which it moves round the axis, so that the direction of the force acting on the particle is also along the tangent to this circle. The moment of the force acting on the particle round the axis of rotation is, therefore, measured by $mra \,.\, r$ or mr^2a.

Now the moments of the forces acting on the particles of the body are all in the same direction, so that the total moment to which the body is subject can be obtained by simply adding together the moments for all the particles of the body. Hence, if $m_1, m_2, m_3, m_4 \ \ldots$ denote the masses of the particles which make up the body, and $r_1, r_2, r_3, r_4 \ \ldots$ denote respectively the distances of these particles from the axis of rotation, the total moment of the forces acting on the body round the axis of rotation is given by—

$$G = m_1r_1^2a + m_2r_2^2a + m_3r_3^2a + \ \ldots$$

or

$$G = a\,[m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ \ldots \ ].$$

The quantity $[m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ \ldots \ ]$ is the sum of all the products obtained by multiplying the mass of every particle in the body into the square of its distance from the axis of rotation. It is called the **moment of inertia** of the body round the axis of rotation.

If this quantity be here denoted by I, we have—

$$G = Ia.$$

This is the relation which corresponds, in the case of motion of rotation, to the relation, $F = ma$, in the case of motion of translation.

It should be noted that G denotes the *moment* of the force acting on the body, taken round the axis of rotation, I denotes the *moment* of inertia of the body round the axis, and a denotes the *angular* acceleration of the body.

46. **Impulse.**—It has now been established that when a body of mass m moves with acceleration a, the force acting on the mass is given by the relation, $F = ma$, in units consistent with those in which m and a are expressed.

It has also been explained in Art. 36 that this relation is equivalent to the statement that the force acting on a body is measured by the time-rate at which the momentum of the body changes. That is, the force acting on a body at any instant is measured by the rate of change of momentum of the body at that instant.

In the case of some forces, however, it is practically impossible to apply this method of measurement. If a force acts on a body for a very short interval of time and, it may be, is not even constant during that interval, it is impossible to determine the rate of change of momentum which it produces in the body at any instant.

For example, when a golf club strikes a ball the time during which the club acts on the ball is so very short that it is practically impossible to determine the force exerted on the ball at any instant by the *rate of change* in its momentum at that instant.

A force of this kind is called an *impulsive force*, and is usually measured, not by the *rate of change* of momentum which it produces, but by the *total change* of momentum which it produces in the body on which it acts. The total change of momentum produced by an impulsive force is called an **impulse.** Thus, if a golf ball of mass m is struck by a club, and leaves the club with a velocity v, the impulse given to the ball is measured by mv, and this is taken as the measure of the whole action of the club on the ball during the stroke.

47. **Impact of Inelastic Bodies.**—If a body of some inelastic material, such as clay, is in motion, and strikes against

another body of the same material at rest, or also in motion, the bodies do not rebound from each other after the *impact*, but adhere together, and, if free to move, move on as one mass.

Suppose an inelastic body, A, of mass m_1, moving with uniform velocity v_1, to overtake another inelastic body, B, of mass m_2 moving in the same direction with velocity v_2; and that after the impact the two bodies move on as one mass with velocity v, in the same direction as before. It will be clear that during the time of impact the one mass acts impulsively on the other in accordance with Newton's second law of motion; the forward impulse communicated by A to B is $(m_2v - m_2v_2)$, or $m_2(v - v_2)$, and the backward impulse communicated by B to A is $(m_1v_1 - m_1v)$, or $m_1(v_1 - v)$, and these two impulses being related as "action" and "reaction" must be equal. That is, we have

$$m_2(v - v_2) = m_1(v_1 - v);$$

or

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v.$$

This result shows that the total momentum of the two bodies is unchanged by their impact. The total momentum of the bodies before impact is $(m_1v_1 + m_2v_2)$, and the total momentum after impact is $(m_1 + m_2)v$, and the relation obtained above shows that these two quantities are equal.

This result is an example of the general principle of **conservation of momentum.** This principle states that the total momentum of an isolated system of bodies is constant, and cannot be changed by any mutual action between the bodies. The principle follows directly from the second law of motion, for if mutual force takes place between any two bodies of the system, the momentum which one gains will be exactly equal to the momentum which the other loses, and the total momentum of the system will remain unchanged by this transfer of momentum from one body to the other.

In applying the principle to simple cases of direct impact between inelastic bodies, it must be remembered that momentum is a vector quantity, and that it is necessary, therefore, to distinguish between momenta in opposite directions by difference in sign.

Examples.—1. An inelastic body of 20 grammes mass, moving with a velocity of 10 cms. per second, meets another inelastic body of 10 grammes mass moving in the opposite direction with a velocity of 35 cms. per second, find the velocity of the combined masses after the impact.

Here, if we take the momentum of the first body of 20 grammes mass to be positive in sign, the momentum of the other body moving in the opposite direction will be negative, and the total momentum before impact is given by

$$\{(20 \times 10) - (10 \times 35)\} \text{ units};$$

or

$$-150 \text{ units (cm. gramme)}.$$

That is, 150 units in the same direction as the momentum of the body of 10 grammes mass. Hence, if v denote the velocity of the combined masses after impact, we have

$$30v = -150,$$

or

$$v = -5.$$

That is, the combined masses move after impact in the same direction as the body of 10 grammes mass, with a velocity of 5 cms. per second.

2. A bullet of 20 grammes mass is fired from a rifle of 4,000 grammes mass, and leaves the barrel with a velocity of 30,000 cms. per second. If the rifle when fired is suspended freely, with its barrel in a horizontal position, find the initial velocity of its recoil.

Here the system considered is made up of three bodies, if we neglect the suspension strings, cartridge case, wads, etc. These three are the rifle, the bullet, and the charge of powder, and of these we may neglect the charge of powder, since no data are given respecting it.

Considering only the rifle and the bullet the momentum before firing is of zero value.

After firing, if we take the momentum of the bullet to be positive, the total momentum of the two bodies is

$$20 \times 30{,}000 - 4{,}000v,$$

where v denotes the initial velocity of the recoil of the rifle.

We, therefore, have

$$600{,}000 - 4{,}000\,v = 0,$$

or,

$$4{,}000\,v = 600{,}000.$$

That is,

$$v = 150.$$

The initial velocity of the rifle in recoil is, therefore, 150 cms. per second.

This question may also be solved by assuming that the charge of powder when exploded gives equal impulses in opposite directions to the bullet and the rifle.

This assumption at once gives

$$4{,}000\,v = 600{,}000,$$

or

$$v = 150, \text{ as before.}$$

CHAPTER X.

WORK AND ENERGY.

48. **Work.**—When the point at which a force acts is displaced, **work** is said to be done, either *by the force* or *against the force.*

If the displacement is along the line of action of the force, then work is done *by the force* if the direction of the displacement is the same as that of the force, and work is done *against the force* if the direction of the displacement is opposite to that of the force.

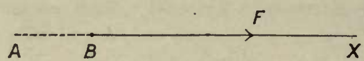

Fig. 52.

Fig. 53.

Thus, let a force F be supposed to act at a point A along AX, and let the point of application be displaced from A to B along the line of action of the force. Then if the direction of the displacement AB is the same as that of the force, as in Fig. 52, work is done by the force. If, however, the direction of the displacement AB is opposite to that of the force, as in Fig. 53, work is done against the force.

If the displacement is not along the line of action of the force, but inclined to it, then work is done by the force or against the force, according as *the component of the displacement along the line of action of the force* is in the same direction as the force, or in the opposite direction.

Thus, let a force F be supposed to act at a point A in a direction AX, and let the point of application be displaced from A to B in a direction inclined to the line of action of the force. Then, if the direction of Ab, the component of the displacement AB, along the line of action of the force is in the same direction as the force, as in Fig. 54, work is done by the force. If, however, the direction of the component Ab is opposite to that of the force, as in Fig. 55, work is done against the force.

The work done by or against a force is measured by the

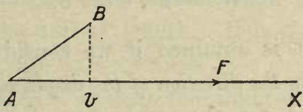

Fig. 54.

product of the magnitude of the force into the displacement or its component along the line of action of the force.

Thus, if the point of application of a force F is displaced through a distance s along the line of action of the force in the

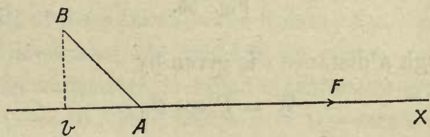

Fig. 55.

same direction as the force, then W, the work done by the force, is given by

$$W = Fs.$$

That is, if a force F *acts through* a distance s, the work done is given by

$$W = Fs.$$

Similarly, if the point of application is displaced through a distance s in the direction opposite to that in which the force acts, the work done against the force is given by $W = Fs$.

If the displacement s be taken as positive when in the same direction as the force, and negative when in the opposite direction, work done by the force will be positive in sign, and work done against the force will be negative in sign.

When the point of application of a force is displaced through a distance s in a direction making an angle θ with the direction of the force, as in Fig. 56, *the component of the displacement in the direction of the force* is $s \cdot \cos \theta$, and the work done by the force is given by

$$W = F \cdot s \cos \theta.$$

The same result is obtained if we consider $F \cos \theta$, *the component of the force in the direction of the displacement* to act through the displacement s, for the work done by the force $F \cos \theta$

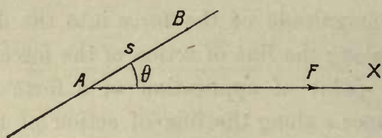

Fig. 56.

acting through a distance s is given by

$$W = F \cos \theta \cdot s.$$

It should be noticed that when the point of application of a force is displaced in a direction *at right angles* to the direction of the force, *no work is done* by or against the force.

If two forces act on a body in opposite directions along the same line, and the body is displaced in the direction of one of the forces, work is done by one force against the other. Thus, if two forces, P and F, act on the body in opposite directions, and the body is displaced through a distance s in the direction of the force P, then the work done by the force P is Ps, and the work done against the force F is Fs.

In this case, if we assume P and F to be the only forces acting on the body, *the work done on the body* is given by ($Ps - Fs$). That is, if we consider work done to be positive when done *by* a force, and negative when done *against* a force *the work done on a body* during any translational displacement of the body is the algebraic sum of the work done by all the forces acting on the body.

Work is a scalar quantity and not a vector quantity. That is, in the measurement of work we have to deal with magnitude only, without reference to direction.

49. **Units of Work.**—The unit of work is derived from the unit of force and the unit of length. It is the work done when unit force acts through unit distance.

Hence, in the C.G.S. system the unit of work is the work done when a force of one dyne acts through a distance of one centimetre. This unit of work is called an **erg.**

The unit of work in the English F.P.S. system is the work done when a force of one poundal acts through a distance of one foot. This unit of work is called a **foot-poundal.**

Work is very commonly expressed in gravitational units, in which the unit of force taken is the weight of unit mass.

The work done when *the weight of one gramme* acts through a distance of one centimetre, is called a **centimetre-gramme**, and is equal at any place to *g ergs*, where g is the acceleration due to gravity at that place. That is, in London a centimetre-gramme is equal to 980·6 ergs. Similarly, the work done when *the weight of one pound* acts through a distance of one foot is called a **foot-pound**, and is equal at any place to *g poundals*, where g is the acceleration due to gravity at that place. That is, in London a foot-pound is equal to about 32·18 foot-poundals.

From these definitions of units of work, it will be seen that the work done against the weight of the mass in lifting a mass of M grammes through a vertical distance of h centimetres is Mh *centimetre-grammes*, or Mhg ergs.

Similarly, the work done in lifting a mass of M pounds vertically through a distance of h feet is Mh *foot-pounds*, or Mhg foot-poundals.

The dyne being a very small unit of force, the erg is a very small unit of work. For example, the work done in lifting this book from the floor on to a table would be something like twenty million ergs. A larger unit, called a **joule**, containing ten million, or 10^7 ergs is, therefore, sometimes used.

It can be calculated from the relations already given that

$$1 \text{ foot-poundal} = 4{\cdot}214 \times 10^5 \text{ ergs},$$

and

$$1 \text{ foot-pound} = 12{,}283 \text{ centimetre-grammes} =$$

$$1{\cdot}356 \times 10^7 \text{ergs} = 1{\cdot}356 \text{ joules}.$$

For rough purposes it may be remembered that a joule, or 10^7 ergs, is nearly three-quarters of a foot-pound.

50. **Energy.**—Energy is capacity for doing work as defined in the foregoing article.

A body may possess energy in virtue of being in motion, for a body in motion is able to do work against an opposing force until it comes to rest.

A body may also possess energy in virtue of the configuration of its parts. A compressed spiral spring, for example, possesses energy in virtue of the configuration which constitutes its compression, for it is able to do work against an opposing force in expanding.

A system of bodies which exert force mutually on each other may, in the same way, possess energy in virtue of the configuration of the system. For, if the system resists any change of configuration impressed on it by the action of an external force, it will be able to do work against an external force in recovering its original configuration.

It will be seen that when a body in motion *does work against a force* acting on the body, it *loses* energy, but *if work is done by* *a force* acting on the body in increasing its momentum, it *gains* energy. In either case *the loss or gain of energy is measured by the work done by or against the force.*

Similarly, when a body, or system of bodies, *does work against an external force* in undergoing a change of configuration, the body, or system of bodies, *loses* energy by the change, but *if work is done by an external force* in producing a change of configuration, then the body, or system of bodies, *gains* energy by the change. In either case *the loss or gain of energy by the body, or system of bodies, is measured by the work done by or against the external force.*

Energy is thus measured as work, and the units employed in its measurement are the same as those employed for the measurement of work.

The energy which a body possesses in virtue of its motion is called **kinetic energy**, and the energy which a body, or system of bodies, possesses in virtue of configuration, is called **potential energy**.

It will be found that in the measurement of energy we are generally called upon to measure the energy which a body, or system of bodies, gains or loses, and not the whole energy which the body, or system of bodies, may possess.

51. **Kinetic Energy.**—As stated above, the energy which a body possesses in virtue of its motion, is called **kinetic energy**.

If a body of mass m moves from rest, with motion of translation under the action of a constant force F, the body moves with uniform acceleration along a straight line in the direction in which the force acts. If the force be allowed to act on the body through a distance s, the work done by the force is Fs, and this is also, as explained above, the measure of the kinetic energy gained by the body.

But if a denote the acceleration of the body, and v the velocity it acquires in moving over the distance s, we know that

$F = ma$, and also, by the relation given in Art. 29, that $v^2 = 2as$. That is,

$$F = ma, \text{ and } s = \frac{v^2}{2a}.$$

It follows, therefore, that $Fs = ma \cdot \frac{v^2}{2a} = \frac{mv^2}{2}$.

That is, the kinetic energy of a body of mass m moving without rotation with a velocity v, is given by $\frac{1}{2} mv^2$. This may be written in the form—

$$K.E. = \tfrac{1}{2} mv^2.$$

If m is expressed in grammes, and v in cms. per second, the kinetic energy is expressed in ergs. Similarly, if m is expressed in pounds, and v in feet per second, the kinetic energy is expressed in foot-poundals.

In the same way if a body of mass m, moving without rotation under the action of a constant force, F, acting in the direction of motion, changes its velocity from u to v in moving over a distance, s, the work done by the force is Fs, and this is also the measure of the kinetic energy gained by the body as it moves over the distance s, and its velocity changes from u to v. But, as above, if a denote the acceleration of the body we know that $F = ma$, and also that $v^2 = u^2 + 2as$, or $v^2 - u^2 = 2as$. That is,

$$F = ma, \text{ and } s = \frac{v^2 - u^2}{2a}.$$

It follows, therefore, that

$$Fs = ma \cdot \frac{v^2 - u^2}{2a} = \frac{m(v^2 - u^2)}{2}.$$

That is, the kinetic energy gained by the body is $\frac{m(v^2 - u^2)}{2}$, or $\frac{1}{2} mv^2 - \frac{1}{2} mu^2$.

This result is in accordance with that obtained above, for if we take the kinetic energy of the mass to be $\frac{1}{2} mv^2$ when moving with a velocity v, and $\frac{1}{2}mu^2$ when moving with a velocity u, the gain of kinetic energy when the velocity increases from u to v is evidently $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$, as obtained above. Similarly, when the velocity of the mass decreases from u to v, the loss of kinetic energy is given by $\frac{1}{2}mu^2 - \frac{1}{2}mv^2$.

Examples.—1. A body of 10 lbs. mass is moving without rotation, with a velocity of 64 ft. per second, find its kinetic energy in foot-pounds.

The kinetic energy of the mass is given by the relation—

$$\text{K.E.} = \tfrac{1}{2}mv^2.$$

That is,

$$\text{K.E.} = \tfrac{1}{2} . 10 \times 64^2$$
$$= 20{,}480.$$

Since m is expressed in pounds, and v in feet per second, the kinetic energy will be in foot-poundals.

The kinetic energy of the mass is, therefore, 20,480 foot-poundals, or, if we take the acceleration due to gravity as 32 feet-per-sec. per sec., the kinetic energy is $\frac{20{,}480}{32}$, or 640 foot-pounds.

2. A mass of 1 kilogramme moving without rotation does work against a constant opposing force through a distance of 1 metre, and in doing this work its velocity is reduced from 500 cms. per sec. to 400 cms. per second; find the amount of work done against the opposing force, and also the magnitude of this force.

From the data of the question, the mass of the body is 1,000 grammes, and its velocity is reduced from 500 cms. per sec. to 400 cms. per sec., in doing work against the opposing force. The kinetic energy lost by the body in doing this work is given, therefore, in ergs by the relation—

$$\text{Loss of K.E.} = \tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2,$$
$$= \frac{m\,(a^2 - v^2)}{2},$$
$$= \frac{1{,}000\,(500^2 - 400^2)}{2},$$
$$= 45{,}000{,}000.$$

That is, the loss of kinetic energy is 45,000,000 ergs, or 4·5 joules.

If the opposing force against which work is done, be denoted by F in dynes, the work done in overcoming this force through a distance of 1 metre, or 100 cms., is $(F \times 100)$ ergs, and we have, therefore,

$$100\,F = 45{,}000{,}000,$$

or

$$F = 450{,}000.$$

That is, the force is equal to 450,000 dynes, or $\frac{450{,}000}{981}$ gramme-weights, if we take the value of g to be 981 cms.-per-sec. per sec. That is, the force is approximately equal to the weight of 458·7 grammes.

It is important to note that the kinetic energy given by $\frac{1}{2}mv^2$ for a body of mass m, moving with a velocity v, is the kinetic energy of the body *relative to a particular point.* The velocity of the body is measured relative to this point, and the displacement of the point of application of any force acting on the body is its displacement relative to this point.

For example, if a body of mass m moves without rotation inside a railway carriage, with a velocity v relative to the carriage, its kinetic energy, or its energy of motion *relative to the carriage*, is given by $\frac{1}{2}mv^2$; if, however, the velocity of this body, relative to a point on the earth's surface, is denoted by V, then its kinetic energy relative to this point is given by $\frac{1}{2}mV^2$. In the one case the body could, in coming to rest relative to the carriage, do $\frac{1}{2}mv^2$ units of work against a force applied by an agent in the carriage and moving with the carriage; in the other case the body could, in coming to rest relative to the point on the earth's surface, do $\frac{1}{2}mV^2$ units of work against a force applied by an agent at this point.

When it is stated that the kinetic energy of a body of mass m, moving without rotation with a velocity v, is $\frac{1}{2}mv^2$, the velocity v is usually understood to mean the velocity of the body relative to a point on the earth's surface at the place where the body is in motion, and the kinetic energy, $\frac{1}{2}mv^2$, is the kinetic energy of the body relative to this point.

A body on the surface of the earth possesses kinetic energy in virtue of its motion as part of the earth. This energy cannot, however, be expended in doing work except against an external force applied by a body or agent external to the earth. That is, the kinetic energy which a body possesses in virtue of its motion as part of the earth is zero relative to the earth, although it may be very great relative to an external point.

52. **Potential Energy.**—It has been explained that a body, or system of bodies, may possess energy in virtue of its configuration, and that energy due to configuration is called **potential energy.**

Thus, when a spring is compressed, work is done against the elastic resistance which the spring offers to compression, and the spring gains energy. Similarly, when the compressed spring expands it is able to do work against an external force opposing its expansion, and so loses energy. The spring thus gains and loses energy by change of configuration, and may, therefore, possess energy in virtue of its configuration.

In the same way a strip of steel, or wood, or any elastic material, gains energy when its configuration is changed by bending it or twisting it, and is able to do work against an external force in recovering its original configuration. It may, therefore, like the spring possess energy in virtue of its configuration.

Change of configuration produced against the resistance offered by the body in virtue of its elasticity* is usually called **strain.** It may, therefore, be said that any elastic body may possess energy of configuration in virtue of any strain which may be set up in it. This form of energy is generally called **energy of strain,** and is a form of potential energy.

The most important case of potential energy which we have to consider, however, is the energy which a system made up of the earth and a body on, or near, its surface, possesses in virtue of its configuration.

The two bodies—the earth and the body near its surface—mutually attract each other, so that if the distance between

* See Chapter xvii.

them is increased, work is done against this mutual force of attraction, and the system gains energy by the change of configuration. For example, if a body on the surface of the earth is raised vertically upwards, so as to increase its distance from the earth, work is done against the force with which the earth attracts it, that is, *work is done against its weight,* and the system made up of the earth and the body gains a quantity of energy equal to the work done in raising the body against its own weight. This gain of energy is due to the change in the configuration of the system; it is sometimes spoken cf as the potential energy gained by the body in virtue of its change of position relative to the earth.

In the same way if a body on the surface of the earth falls vertically downwards, so as to decrease its distance from the earth, *work is done by its weight*, and the system made up of the earth and the body loses potential energy equal to the work done. This loss of energy is due to the change in the configuration of the system, and is commonly spoken of as the potential energy lost by the body in virtue of its change of position relative to the earth.

It should be noticed in this case that if the falling body does no work against an external force, the potential energy it loses is expended in doing work on the body itself by setting it in motion, and thereby produces an amount of kinetic energy exactly equal to the potential energy lost. If, however, the falling body does work against an external force, the potential energy it loses is wholly or partially expended in doing this work, and is thereby wholly or partially transferred to another body, or system of bodies.

When the potential energy lost by the body is only partially expended in doing work against an external force, the remainder is expended in setting the body in motion, and thereby produces, as explained above, an equal amount of kinetic energy.

For example, if two masses, A and B, of which A is the greater, are slung by a string over the wheel of an Atwood's machine, the weight of A in falling does work against the weight of B in raising it. The potential energy lost by A in falling is thus expended, partly in doing work against the weight of B, and partly in setting A and B in motion. The portion expended in doing work against the weight of B produces an increase in B's potential energy, equal to the amount so expended, and the portion expended in setting A and B in motion produces an amount of kinetic energy also equal to the amount so expended. The potential energy lost by A in falling is thus transferred, partly to B as increase of its potential energy, and partly to A and B as kinetic energy.

The potential energy which a body is said to possess, in virtue of its position relative to the earth, is really the potential energy which the system made up of the body and the earth possesses in virtue of its configuration. It is, however, convenient to consider the energy as the energy of the body, and it is usually called the **gravitational potential energy** of the body.

It has been shown that when a body of mass m is raised through a vertical distance h at any place, the work done against its weight is measured by mgh, where g denotes the acceleration due to gravity at the place.* The potential energy gained by the mass will, therefore, also be measured by mgh. That is, when a body of mass m is raised through a vertical distance h at any place, the increase in the gravitational potential energy of the body is equal to the work done against the weight of the body in raising it, and is given by mgh, where g denotes the acceleration due to gravity at the place. Similarly, when a body of mass m falls through a vertical distance h at any place, the decrease in the gravitational potential energy of the body is equal to the work done by the weight of the body in falling, and is

* The distance h is supposed to be so small, compared with the radius of the earth, that the weight of the body may be assumed to be constant over the whole distance.

measured by mgh, where g denotes the acceleration due to gravity at the place.

For example, if a body of 10 pounds mass is raised through a vertical distance of 10 feet, at a place where the acceleration due to gravity is 32·18 feet-per-sec. per sec., the increase in the gravitational potential energy of the body is $10 \times 10 \times 32{\cdot}18$ foot-poundals, or 3218 foot-poundals. The increase in the potential energy of the body may also be expressed as (10×10), or 100 foot-pounds, a foot-pound in this case being equal to 32·18 foot-poundals.

Similarly, if a body of 500 grammes mass falls through a vertical distance of 100 cms. at a place where the acceleration due to gravity is 981 cms.-per-sec. per sec., the decrease in the gravita-

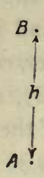

Fig. 57.

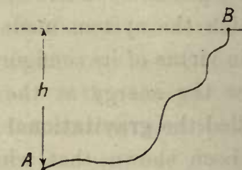

Fig. 58.

tional potential energy of the body is $500 \times 100 \times 981$ ergs, or 49,050,000 ergs or 4·905 joules. The decrease in the potential energy of the body may also be given as (500×100), or 50,000 centimetre-grammes, a centimetre-gramme in this case being equal to 981 ergs.

It should be noticed that the work done in raising a body from a point A to a point B through a vertical distance h is the same whether the body is raised directly along the vertical at A, as in Fig. 57, or follows any other path through the same vertical distance, as in Fig. 58. The weight of the body acts *vertically*, so that the work done is given by the product of the weight into the *vertical* displacement, whatever may be the actual path of displacement.

53. **Forms of Energy.**—The energy which any body, or system of bodies, possesses can always be classified as kinetic energy, or potential energy, or a combination of these two general forms of energy.

There are, however, a number of special cases or forms of energy which should be noticed.

Thus, kinetic energy, or energy of motion, presents two important cases in the energy of a body in motion without rotation, and the energy of a body in rotation round an axis. The first of these two cases is dealt with in Art. 51, the second in Art. 59.

The energy of a body in **vibratory motion** is a combination of kinetic energy and potential energy. For example, the energy of the bob of a simple pendulum in vibratory motion (Art. 34) is, in general, partly kinetic energy, and partly gravitational potential energy. The energy which the bob possesses at any instant during its vibration in *excess* of the energy it possesses when at rest, is the energy which the body possesses as energy of vibration. When the bob is at either of the extreme points in its path this energy is wholly gravitational potential energy, and when the bob is at the lowest point in its path the energy is wholly kinetic energy, but at any other point in its path the energy is partly potential and partly kinetic. As the bob falls from either of the highest points in its path to the lowest point, it loses potential energy and gains kinetic energy, and as it passes from the lowest point to either of the highest points, it loses kinetic energy and gains potential energy. The total energy of the bob thus remains constant, except in so far as it is diminished by doing work against external forces, such as air resistance or friction.

The energy of **wave motion** * is another form of energy which is partly kinetic and partly potential. The particles of the

* See Chapter iii. in Part II. on *Sound*, and Chapter i. in Part III. on *Light*.

medium through which the wave motion is propagated are in vibratory motion, and the medium itself is subject to strain; the energy of the motion must, therefore, be in part kinetic energy, and in part potential energy of strain. **Sound** * may be considered as longitudinal wave motion in material media, so that sound as a form of energy is energy of longitudinal wave motion in material media. **Light** * is transverse wave motion in the ether, and light as a form of energy is, therefore, the energy of transverse wave motion in the ether.

Any piece of matter may be considered as a system of molecules held together by the action of intermolecular forces. The piece of matter may, therefore, possess *molecular potential energy* in virtue of its molecular configuration or *state of aggregation*, and as the molecules are not at rest but in motion, it may also possess *molecular kinetic energy* in virtue of the motion of its molecules.

It is generally agreed that the molecular kinetic energy of any body constitutes the heat of the body. **Heat** as a form of energy must, therefore, be considered as molecular kinetic energy.

The molecular potential energy of a body cannot be so definitely labelled. It is known, however, that when a substance changes its state of aggregation from the solid state to the liquid state, or from the liquid state to the vapour state, it absorbs a quantity of energy in the form of heat, which is in part expended in supplying the increase of molecular potential energy which attends the change of state. Thus a gramme of ice at 0° C. absorbs a quantity of heat equivalent to 336 joules of energy in changing into a gram of water at 0° C., and thus provides for the increase of molecular potential energy which attends the change from solid ice to liquid water.

The heat thus absorbed at change of state is known as **latent**

* See Chapter iii. in Part II. on *Sound*, and Chapter i. in Part III. on *Light*.

heat. It may be noticed that since a substance always absorbs latent heat in changing from the solid state to the liquid state, and from the liquid state to the vapour state, it follows that the molecular potential energy of a substance is greater in the liquid state than in the solid state, and greater still in the vapour state than in the liquid state.

Just as a piece of matter may be considered as a system of molecules held together by intermolecular forces, so a molecule may be considered as a system of atoms held together by interatomic forces. A molecule may, therefore, possess potential energy in virtue of its atomic configuration, and atomic kinetic energy in virtue of the motion of its atoms.

The atomic energy of molecules constitutes what is known as **chemical energy.** Thus, in the chemical reaction represented by the equation—

$$Zn + H_2SO_4 = ZnSO_4 + H_2,$$

the atomic energy of the molecules, $Zn + H_2SO_4$, is greater than the atomic energy of the molecules, $ZnSO_4 + H_2$, and the loss of chemical energy which attends the reaction is evolved as heat.

Recent research goes far to show that atoms are not, as they were thought to be, the ultimate particles of matter, but are themselves more or less complicated systems of particles. On this view, therefore, an atom may possess potential energy in virtue of the configuration of its constituent particles, and kinetic energy in virtue of the motion of these particles. The phenomena of **radioactivity** are supposed to be manifestations of this form of energy.

All the forms of energy referred to above, with the exception of light energy, are associated with matter as energy of motion, or energy of configuration of some material system. It is conceivable, however, that energy is associated in similar ways with ether. The energy of light and radiation generally is known to be energy of transverse wave motion in the ether; electrical and magnetic phenomena are manifestations of different forms of energy of strain and motion in free ether, and in ether associated with matter; and gravitational potential energy is probably a form of energy in the ether.

54. **Conservation of Energy.**—It will be seen from what has been said above that when energy is lost or expended in doing work, an equal amount of energy is gained or produced as the equivalent of the work done. Thus, a body, or system of bodies, may lose energy in one form, and gain an equal amount of energy in some other form; or a body, or system of bodies, may lose energy by doing work on some other body, or system of bodies, which thus gains an equal amount of energy as the equivalent of the work done on it.

When a quantity of energy is lost or expended in this way in one form, and is gained or produced in some other form, it is said to be *transformed*, or to undergo *transformation*, but whatever the nature of the transformation may be, the quantity of energy produced is always equal to the quantity expended or lost.

Thus, when a body falls freely through any distance it loses an amount of gravitational potential energy equal to the work done by the weight of the body during the fall, and gains an amount of kinetic energy also equal to the work done by the weight; that is, the body loses a quantity of gravitational potential energy, and gains an exactly equal quantity of kinetic energy.*

Similarly, in the case of the bob of a simple pendulum in vibration: the bob loses gravitational potential energy, and gains kinetic energy in falling, and it loses kinetic energy and gains gravitational potential energy in rising. In moving over any

* It would be more correct to say that the system made up of the earth and the body loses a quantity of gravitational potential energy (the energy of the configuration of the system), and the masses of the system gain an equal quantity of kinetic energy.

portion of its path, however, the energy lost in one form is exactly equal to the energy gained in the other form, each being equal to the work done by or against the weight of the bob. The vibration energy of the bob thus remains constant,* but is subject to periodic transformation from potential energy to kinetic energy, and from kinetic energy to potential energy.

In the same way when a body in falling raises another body, the potential energy lost by the falling body is equal to the potential energy gained by the body raised, together with the kinetic energy gained by the two bodies. The total energy of the two bodies considered as one system thus remains constant.

In general, therefore, it may be stated that when a body, or system of bodies, A, does work on another body, or system of bodies, B, the body or system A loses energy in some form, and the body or system B gains energy either in the same form or in some other form or forms, and the energy lost by A is exactly equal to the energy gained by B. The energy which A loses is equal to the work done by the force which A exerts on B, and the energy which B gains is equal to the work done against the force which B exerts on A; and as these two forces are equal and opposite in direction by Newton's second law of motion, it follows that the work done by the one must be equal to the work done against the other, and the energy lost by A must, therefore, be equal to the energy gained by B.

Hence, if we suppose a number of bodies, or systems of bodies, to be in dynamical communication with each other, but to be completely isolated from all other bodies, it is evident that the total amount of energy in the complete system must remain constant and unchangeable. Any portion of the energy of the system may undergo any of the transformations of which energy is capable, or may be transferred from any part of the system to

* The work done against air resistance, etc., is here neglected. See below.

another part, but whatever quantity of energy disappears in one form an exactly equal quantity reappears in another form; or whatever quantity disappears at any point in the system an exactly equal quantity reappears at another point. That is, the total quantity of energy in the complete system is constant and unchangeable, and cannot be increased or diminished in any way.

This is the principle known as the principle of **conservation of energy.** Maxwell states this principle in the following form.

The total energy of any material system can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.

It should be noticed that in this statement the words, "by *any action between the parts of the system*," implies that the system is supposed to be free from the action of all bodies external to itself. If any external body were allowed to do work on the system it would gain energy, and if the system were allowed to do work on any external body it would lose energy, but if the system, however complex or extensive it may be, is completely isolated from all external bodies, it can neither lose nor gain energy.

If we consider the total universe, of which we have cognisance to be isolated from the action of all bodies external to it, the principle of conservation of energy states that the total quantity of energy in this universe is constant and unchangeable.

The transformations which energy undergoes in the world around us are almost endless in their variety.

Instances have already been given in which gravitational potential energy is transformed into kinetic energy, and kinetic energy into potential energy. Other instances will readily occur to the reader. When a body is falling freely its gravitational potential energy is being transformed into kinetic energy, and when the body strikes the ground and is brought to rest, its kinetic energy is suddenly transformed into an equivalent quantity of heat.

When a bullet strikes an iron target and is flattened against it, work is done against molecular forces in flattening the bullet, and the kinetic energy of the bullet is thereby converted into heat in the bullet. When a bullet strikes a piece of wood and penetrates it, work is done by the bullet against molecular forces and against friction, and the kinetic energy of the bullet is transformed into heat. This heat is primarily produced where the work is done in the shattered wood and at the surface of the bullet, but much of it passes by conduction into the bullet.

A simple pendulum in vibratory motion ultimately comes to rest, because the energy which it possesses is gradually expended in doing work against air resistance and friction, and also to a small extent against molecular friction in the suspension thread at the point where bending takes place. The energy of vibratory motion which the pendulum possesses is in this way transformed into heat in the air and in the thread.

The pendulum of a clock continues in motion because it receives a regular supply of energy from the spring or weights of the clock. If the clock is allowed to run down the pendulum soon comes to rest in doing work against frictional resistance, and its energy is thereby transformed into heat.

When a tuning fork is set in vibration it continues in vibration for some time, but ultimately comes to rest. During vibration its energy is, as in all cases of vibration, subject to periodic transformation from potential energy to kinetic energy, and from kinetic energy back again to potential energy, the potential energy in this case being potential energy of bending strain in the prongs of the fork.

As the vibration continues, however, the stock of energy which the fork possesses is gradually expended in doing work against molecular friction in bending the prongs, and also in setting up longitudinal wave motion, or sound waves, in the surrounding medium. The energy expended in doing work against molecular friction in bending the prongs, is transformed into heat in the prongs. The energy expended in setting up wave motion in the surrounding medium travels out from the fork into the medium as energy of wave motion or sound, and is ultimately expended in doing work against molecular friction in the medium and thereby transformed into heat in the medium.

When the sun shines on the earth and warms it, the energy of solar radiation—that is, the energy of transverse wave motion in the ether—is transformed into heat in the earth. Energy is in this way transferred from the sun to the earth.

When a piece of coal burns in air the chemical energy liberated by the combination of the carbon and hydrogen of the coal with the oxygen of the air is transformed into the heat evolved by the combustion of the coal.

When the terminals of a voltaic cell, such as a Bunsen cell, are joined by a wire, the chemical energy liberated by the chemical reaction in the cell is transformed into electrical energy, and an electric current is produced in the circuit of the cell and the wire joining its terminals, and this energy is in turn transformed into heat in the circuit through which the current passes.

In all cases where work is done by muscular effort chemical action goes on in the muscular tissue, and the action is of such a nature that the tissue loses chemical energy. This energy is expended in doing work, and may thereby be transformed into other forms of energy. Thus, when a man lifts a body he does work against its weight, and the chemical energy expended in the muscles in doing this work is transformed into an increase in the gravitational potential energy of the body.

When a *heat engine* of any kind is in action, a proportion of the heat produced by the combustion of the fuel in the engine is expended in doing work, and is thereby transformed into some other form of energy. For example, in the case of a railway engine, the heat expended in setting a train in motion is transformed into the kinetic energy of the moving train. Similarly, in the case of a gas engine used to drive a dynamo, the heat expended by the engine in driving the dynamo is transformed into the electrical energy produced by the dynamo.

It is important to notice that although all other forms of energy are readily transformed into heat, heat itself can be transformed into other forms of energy only under certain special conditions, such as obtain in a heat engine. It is found that heat can be expended in doing work (and thereby transformed into some other form of energy) only in passing from one body, A, to **a colder body,** C, through an intermediate working substance B. Of the heat which passes from A to B, a portion must pass from B to C, but the remainder may, by the action of B, be expended in doing work. For example, in a *steam engine,* the steam as working substance (B) takes heat from the boiler (A) and gives up heat to the condenser (C) but it receives more heat from the boiler than it gives up to the condenser, and by its action in the cylinder on the mechanism of the engine it expends the difference in doing work. In any heat engine a large portion of the heat produced by the combustion of the fuel is lost by conduction, another portion is passed on to the condenser, and a considerable portion is expended in doing work against friction in the mechanism of the engine, and is thereby converted into heat in the working parts of the engine. It follows that the proportion of the heat produced in the engine which is transformed into some other form of energy is comparatively small.

It will be seen from what has been said that difference of temperature is essential for the transformation of heat into any other form of energy. Hence, if a system possesses energy in the form of heat only, and if the temperature of the system is uniform throughout, no portion of the energy of the system can undergo transformation of any kind. That is, the system cannot expend any of its energy in doing work, or the energy of the system is not *available* for doing work.

55. **Dissipation of Energy.**—Among the many transformations of energy which go on around us, it will be noticed that energy in every form is continuously undergoing transformation into heat which passes by conduction into the earth. This heat tends, by the process of conduction, to distribute itself, without difference of temperature throughout the mass of the earth; in this form it merely adds to the store of heat energy in the earth, and is not available for retransformation into any other form of energy. This process must be going on everywhere in the material universe where energy exists in more than one form. It follows, therefore, since the total quantity of energy in the material universe is constant and unchangeable, that the supply of energy available for doing work is steadily decreasing by transformation into heat which cannot be expended in doing work.

The material universe is thus tending steadily to a state in which the whole of the energy it possesses will be in the form of heat, and the temperature throughout the system will be uniform.

In this state no part of the energy of the universe can be expended in doing work, and all the processes associated with doing work by the action of force will be at an end.

56. **Power or Activity.**—Power or activity is the time-rate at which work is done. Thus, if a quantity of work W is done in a time t, then W/t is the average *power* or *activity* exerted. For example, if a mass of 1 ton is raised through a vertical distance of 120 feet in 7 minutes, the average power exerted is $\frac{2{,}240 \times 120}{7 \times 60}$ foot-pounds per second, or 640 foot-pounds per second.

In English units power is usually measured in horse-power,

one horse-power being defined as a rate of doing work equal to 33,000 foot-pounds per minute, or 550 foot-pounds per second.

Example.—A pumping engine pumps water from a depth of 400 feet at the rate of 120,000 gallons per hour. If the water leaves the pump with a velocity of 2 feet per second, find in horse-power the rate at which work is done on the water in raising it.

The mass of water raised per minute is

$$\frac{120,000 \times 10}{60} \text{ pounds,}$$

or 20,000 pounds.

The work done against the weight of this mass of water in raising it is

$$20,000 \times 400 \text{ foot-pounds,}$$

or 800,000,000 foot-pounds.

The work done in giving this mass of water the kinetic energy with which it leaves the pump is

$$\frac{20,000 \times 4}{2 \times 32} \text{ foot-pounds,}$$

or 1,250 foot-pounds, if we take the acceleration due to gravity to be 32 ft.-per-sec. per second.

The total work done per minute on the water is, therefore 8,001,250 foot-pounds, and the power expended is

$$\frac{8,001,250}{33,000} \text{ horse-power,}$$

or, 242·462 horse-power.

In C.G.S. units power is usually measured in ergs per second. In electrical measurements a unit of power called the **watt** is generally used, and is defined as one joule per second, or 10^7 ergs per second. A multiple of this unit, known as the **kilowatt** and equal to 1,000 watts, is also used. The French *Force de cheval*, the equivalent of the English horse-power, is equal to 75 kilogramme-metres per second, or about 736 watts. A horse-power is equal to 746 watts.

We have, therefore, the following units of power or activity in the English and C.G.S. systems of units.

1 Horse-power = 33,000 foot-pounds per minute = 550 foot-pounds per second.
1 Watt = 1 joule per second = 10^7 ergs per second.
1 Kilowatt = 1,000 watts.
1 Force de cheval = 75 kilogramme-metres per second = 736 watts.

The relation between any of these units may be determined by taking

1 Horse-power = 746 watts = ·746 kilowatt.

That is, three kilowatts is very nearly equal to four horse-power, or a horse-power is approximately equal to three-quarters of a kilowatt.

57. **Relation between Energy and Force.**—If a force F act on any body, or system of bodies, and effects a very small displacement δ in the direction in which it acts, the energy gained by the body, or system of bodies, is equal to the work done by the force and is measured by $F\delta$. Hence, if ε denote the energy gained by the body, or system of bodies, we have

$$\varepsilon = F\delta,$$

or

$$F = \frac{\varepsilon}{\delta}.$$

That is, the force acting on the body or system of bodies in any direction is measured by the rate at which the energy of the body, or system of bodies, changes per unit length for a small displacement in that direction.

A simple case of the application of this principle is given in Art. 51. Here a body moving under the action of a *constant* force F, gains kinetic energy, and the kinetic energy gained in moving over a distance s * from rest is measured by Fs.

Hence, if E denote the energy thus gained, we have

$$E = Fs$$

or

$$F = \frac{E}{s}.$$

That is, the force F is measured by the rate at which the body gains energy per unit distance along the line of displacement.

Since force can be measured in this way it has been called the distance-rate or *space-rate* of change of energy.

58. **Work Done by a Variable Force.**—It has been shown in Art. 51 that the work done by a constant force F in acting through a distance s in the direction in which the force acts is measured by Fs. That is, we have

$$W = Fs.$$

If the force is not constant but variable then the value of F changes from point to point over the distance s, and the relation $W = Fs$ cannot be used. If, however, we imagine the whole displacement s to be divided into a very large number, n, of equal elements or steps, each denoted by δ, and if we assume the force to vary from step to step, but to be constant over each step, the total work done by the force in acting through the distance s is given by

$$W = F_1\delta + F_2\delta + F_3\delta \quad . \quad . \quad . \quad + F_n\delta,$$

where F_1, F_2, F_3, . . . F_n denote the magnitudes of the force over the successive steps of the displacement.

We, therefore, have

$$W = (F_1 + F_2 + F_3 \quad . \quad . \quad . \quad + F_n)\delta,$$

* In this case the force is *constant* so that the displacement s through which the force acts need not be assumed to be very small for the work done by the force is measured by Fs whether s be large or small.

and since $\delta = \frac{s}{n}$ we get

$$W = \frac{(F_1 + F_2 + F_3 \quad . \quad . \quad . \quad + F_n)}{n} . s.$$

If n is very great (infinitely great), δ is so small that the assumption made above is true and the result here obtained gives the true value of the work done.

Now, when n is very great, $\frac{F_1 + F_2 + F_3 \quad . \quad . \quad . \quad + F_n}{n}$ is evidently the *average force*, which acts over the distance s, and if we denote it by $\overline{F}$ we have

$$W = \overline{F}s.$$

If the force varies from point to point over the distance s in a manner which is known and can be expressed mathematically, the work done by the force can be calculated by methods similar in principle to that indicated above. This mathematical method is, however, only possible in certain cases and generally requires a knowledge of advanced mathematical methods.

The simplest general method of determining the work done in a case of this kind is a *graphical method* similar to that explained in Art. 29 for the determination of the space passed over by a body moving with variable velocity. If the manner in which the force varies over the distance s is known it can be represented by a curve of which the abscissa at any point represents the displacement effected in the direction in which the force acts, and the ordinate represents the corresponding value of the force.

Let CD in Fig. 59 represent a curve plotted in this way, so that the ordinate at any point represents the magnitude of the force when the displacement has the value represented by the abscissa of the point. For example, AC represents the magnitude of the force when the displacement is represented by

OA, and BD represents the magnitude of the force when the displacement is represented by OB.

Consider the work done by the force in acting through any very small element of displacement, such as that represented by *ab*. If the force remained constant throughout the small displacement at the value represented by *ac*, the work done would be represented by the area of the rectangle *abce*. For, since *ac* represents the force, and *ab* the displacement considered, the product of the force into the displacement must be represented* by the area of the rectangle contained by *ac* and *ab*. Similarly, if the force had the value represented by *bd* throughout

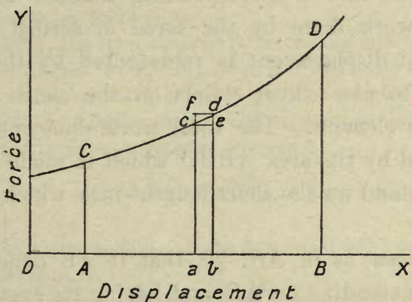

Fig. 59.

the displacement the work done would be represented by the area of the rectangle *abdf*.

Now the work done by the force in acting through the small distance represented by *ab* must be less than that represented by *abdf* and greater than that represented by *abec*, and the difference between each of these two extreme values and the work represented by the strip *abdc* can be made as small as we please by making *ab* small enough. Hence, when *ab* is infinitely small, the work done by the force in acting through the infinitely small element of displacement represented by *ab*, is

* Compare Art. 29.

represented by the area of the strip *abcd* which stands on *ab* as its base, and is bounded along *cd* by the portion of the curve intercepted between the ordinates *ac* and *bd* which form its sides.

It follows from this that if we consider the force to act through any finite displacement, such as that represented by AB, the work done by the force must be represented by the area ABDC which lies between the ordinates AC and BD, and is bounded by the curve along CD. The displacement represented by AB may be divided into a very large number of small equal elements similar to that represented by *ab*, and if AB be divided into a corresponding number of short equal lengths, the work done by the force in acting through each element of the displacement is represented by the area of the strip similar to *abec* which stands on the short length which represents the element. The total work done must, therefore, be represented by the area ABDC which is made up of all the strips which stand on the short lengths into which AB has been divided.

It will be seen, as in Art. 29, that if AE denote the height of a rectangle standing on AB, and having its area equal to the area of ABDC, then AE is the mean or average of the ordinates between A and B, and represents the average force denoted by $\overline{F}$ above.

Example.—It is found by experiment that the force required to stretch a spiral spring is proportional to the elongation produced, and that the force required to stretch it 1 cm. is equal to the weight of 200 grammes; find the work done in stretching the spring so as (*a*) to produce an elongation of 20 cms.; (*b*) to increase the elongation from 20 cms. to 30 cms.

(*a*) Since the force applied in stretching the spring is proportional to the extension produced, the force applied in producing an elongation of 20 cms. increases from its initial zero value to a final value of 20 × 200 gramme-weights or 4,000 gramme-weights, and as it increases *uniformly* with the elongation, the average force exerted in producing the elongation is given in gramme-weights by

$$\bar{F} = \frac{0 + 4{,}000}{2} = 2{,}000.$$

That is, the average force is 2,000 gramme-weights, or (2,000 × 981) dynes.

The work done in producing the elongation is given by

$$W = \bar{F}s,$$

where s is the elongation or the distance through which the force acts. W is therefore given in ergs by

$$W = 2{,}000 \times 981 \times 20,$$

or

$$W = 39{,}240{,}000.$$

That is, the work done in producing an elongation of 20 cms. is 39,240,000 ergs, or 3·924 joules.

(*b*) Similarly, the force applied in increasing the elongation of the spring from 20 cms. to 30 cms. increases *uniformly* with the elongation from an initial value of 4,000 gramme-weights to a final value of (30 × 200), or 6,000 gramme-weights, and the average value of the force exerted is given in gramme-weights by

$$\bar{F} = \frac{4{,}000 + 6{,}000}{2} = 6{,}500.$$

That is, the average value of the force is 5,000 gramme-weights, or (5,000 × 981) dynes.

The work done in increasing the elongation is given by

$$W = \bar{F}s.$$

W is, therefore, given in ergs by

$$W = 5{,}000 \times 981 \times 10,$$

or

$$W = 49{,}050{,}000.$$

That is, the work done in increasing the elongation from 20 cms. to 30 cms. is 49,050,000 ergs, or 4·905 joules.

This example may also be worked by the graphical method explained above. The reader will find it a useful exercise to work it out for himself.

59. **Kinetic Energy of Rotation.**—The kinetic energy of a *particle* of mass m moving with a velocity v is $\frac{1}{2}mv^2$. The kinetic energy of a *body* of mass m moving with motion of

translation with velocity v is also measured by $\frac{1}{2} mv^2$, for every particle in it is moving with the same velocity. The kinetic energy of a body in rotation is not, however, given by this relation, for the linear velocity of motion is not the same for every particle in it. When a rigid body rotates round an axis with angular velocity ω, the velocity of any particle in the body at a distance r from the axis is $r\omega$. If the mass of the particle is denoted by m, the kinetic energy of the *particle* is $\frac{1}{2} m(r\omega)^2$, or $\frac{1}{2} mr^2\omega^2$, and the total kinetic energy of the rotating body is the sum of the kinetic energy of all its particles. Hence, if $m_1, m_2, m_3, m_4, \ldots$ denote the masses of the particles which make up the body, and $r_1, r_2, r_3, r_4, \ldots$ denote respectively the distances of these particles from the axis of rotation, the total kinetic energy of the body is given by

$$E = \tfrac{1}{2} m_1 r_1^2 \omega^2 + \tfrac{1}{2} m_2 r_2^2 \omega^2 + \tfrac{1}{2} m_3 r_3^2 \omega^2 + \ldots,$$

or

$$E = \tfrac{1}{2} \omega^2 [m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots].$$

The expression here given in square brackets is seen to be the moment of inertia of the body round the axis of rotation, as explained in Art. 45, so that if I denote this moment of inertia, we have

$$E = \tfrac{1}{2} I\omega^2.$$

A body moving in a circle with angular velocity ω, is in rotation round an axis passing through the centre of the circle, and at right angles to the plane of the circle. Hence, if I denote the moment of inertia of the body round this axis, its kinetic energy of rotation is given, as above, by

$$E = \tfrac{1}{2} I\omega^2.$$

CHAPTER XI.

COMPOSITION AND RESOLUTION OF FORCES.

60. **Force is a Vector Quantity.**—A force possesses magnitude and direction, and is, therefore, a vector quantity. Forces may, therefore, be compounded and resolved by the rules which apply to the composition and resolution of vector quantities.

In order to specify a force completely it is necessary to specify not only its magnitude and direction, but also its *point of application.* For example, it is not a sufficient specification to state that a force is 100 dynes in magnitude, and acts vertically downwards; the particular point at which it is applied to the body on which it acts, must also be specified. A force may in general be supposed to act at any point in its line of action, but in certain cases it is necessary to specify the exact point at which it acts.

61. **Composition of Forces Acting at a Point.**—If two forces act at a point in the same direction along the same straight line, their resultant is obviously equal to their sum. If they act in opposite directions along the same straight line, their resultant is equal to their difference, and acts in the same direction as the greater force.

Hence, if any number of forces act at a point along the same straight line, their resultant is equal to the difference between the sum of those acting in one direction along the line and the sum of those acting in the opposite direction, and acts in the direction of the greater sum.

If forces acting in one direction along the line are considered positive, and those acting in the opposite direction negative, the resultant of any number of forces acting along the line is the algebraic sum of the forces.

If two forces act at a point in directions inclined to each other, their resultant may be at once obtained by means of the parallelogram or triangle rule for the composition of vector quantities.

Thus, if two forces of magnitude P and Q act at a point O (Fig. 60), and if lines AB and AC are drawn from a point A to represent these forces in magnitude and direction, the resultant of the two forces is represented in the same way by the diagonal AD of the parallelogram ABCD, constructed on AB and AC as

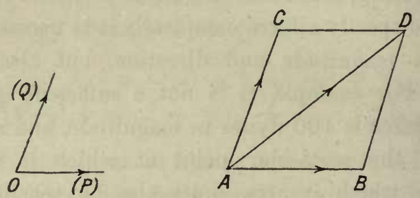

Fig. 60.

adjacent sides. Or, if we apply the triangle rule, and draw *in order* from the point A two lines, AB and BD, to represent the forces P and Q in magnitude and direction, the resultant of the two forces will be represented by the line AD, drawn from the starting point A to the finishing point D.

The magnitude of the resultant can be determined graphically, or calculated from the geometry of the representative parallelogram or triangle as explained in Art. 9. Thus, it can be shown from the geometry of the parallelogram ABDC in Fig. 60 that

$$AD^2 = AB^2 + BC^2 + 2\,AB \,.\, BC \cos BAC.$$

That is, if the magnitude of the resultant force represented by AD is denoted by R, we have

$$R^2 = P^2 + Q^2 + 2\,PQ \cos a,$$

where a denotes the angle between the directions of the two forces of magnitude P and Q.

In the particular case where the angle between the direction of the two forces is a right angle ($a = 90°$), we have the simpler relation—

$$R^2 = P^2 + Q^2.$$

This relation may be obtained directly from the figure by applying *Euc.* i., 47, or deduced from the general relation given above, for, when $a = 90°$ $\cos a = 0$, and the term $2\,PQ \cos a$ disappears from the general relation.

The application of the parallelogram rule to the composition of two forces acting at a point, gives rise to the theorem generally known as the **parallelogram of forces.** This theorem may be stated generally in the following terms.

If two forces acting at a point are represented in magnitude and direction by two straight lines drawn from any point, and a parallelogram be constructed with these two lines as adjacent sides, the resultant of the two forces will be represented by the diagonal of the parallelogram drawn from the point from which the lines representing the forces are drawn.

The truth of this theorem may be verified experimentally by the following experiment.

Experiment 5.—Take any three masses, P, Q, and R, of which P and Q together are greater than R. Attach a length of fine pulley cord to each mass and tie the ends of the three cords in a knot. The cord attached to R should be shorter than those attached to P and Q, which should be fairly long. Set up two good pulleys with their wheels in the same plane, pass the cords carrying the masses P and Q over these pulleys, and let the three connected masses P, Q, R, set in equilibrium in the position shown in Fig. 61, with the knot connecting the three cords at A.

The three forces now acting at the point A are (1) the tension in the cord AL equal to the weight of the mass P; (2) the tension in the cord AM equal to the weight of the mass Q; and (3) the tension in the cord AR equal to the weight of the mass R.

Since these forces exactly balance each other in equilibrium, the weight of the mass R must be equal and opposite to the resultant of the weights of the masses P and Q. That is, the resultant of the weight of P acting along AL, and the weight of Q acting along AM, is equal to the weight of R, and acts vertically upwards. Hence, if the weights of the masses P, Q, and R, be denoted by P, Q, and R respectively, the experiment shows that the resultant of a force of P units acting at A along AL, and a force of Q units acting at A along AM, is a force of R units acting vertically upwards at A.

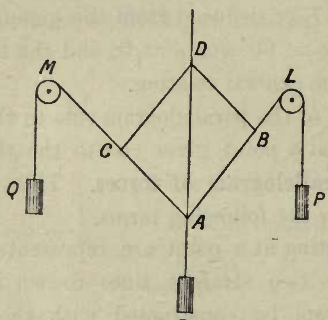

Fig. 61.

Now, bring a sheet of paper pinned on a drawing board close up to the plane in which the cords AL, AM, and AR hang, and fix it in this position. Mark the position of the point A on the paper, and draw lines AL, AM, and AR, marking the directions of the cords AL, AM, and AR. Along the line AL mark off a length AB, P units in length to represent the force P in magnitude and direction, and along the line AM mark off a length AC, Q units in length to represent the force Q in magnitude and direction; then complete the parallelogram ABCD, and draw the diagonal AD.

The diagonal AD represents, in accordance with the parallelogram of forces, the resultant of the two forces P and Q acting along AL and AM respectively.

If the resultant thus represented is the same as that obtained by the experiment, the direction of AD on the paper should be vertical in a line with AR, and it should be R units in length on the same scale as that on which AB represents the force P and AC the force Q.

It will be found that this is the case.

The parallelogram rule for the composition of forces is thus shown to give results in agreement with experiment.

When any number of forces act at a point in different directions, their resultant may be found by the continued application of the parallelogram rule, or by the polygon rule for the composition of vector quantities.

Thus, if forces of magnitude P, Q, R, S and T act at a point O, Fig. 62, and lines AB, BC, CD, DE, and EF be drawn

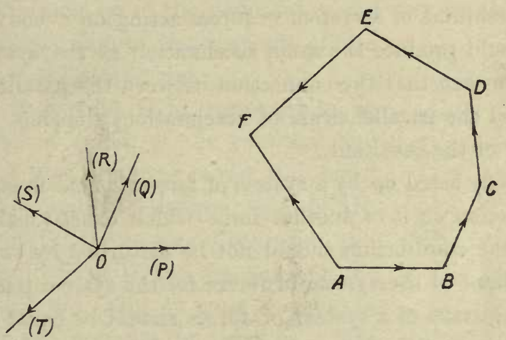

Fig. 62.

in order from a point A to represent these forces, the line AF drawn from the starting point A to the finishing point F, represents the resultant of these forces in magnitude and direction.

It should be noticed in applying the polygon rule for the composition of a number of forces acting at a point (a) that the forces may be taken in any order; and (b) that the forces may act in any direction from the point, and need not all be in one plane.

If the forces are not all in the same plane the "polygon" will not be a plane figure but an irregular figure with sides in different planes, but if the forces all lie in the same plane, it will be a plane polygon.

In either case the form of the figure for a given system of forces will depend upon the order in which the forces are taken for representation, but the closing line representing the resultant will be the same in length and direction whatever the order may be.

62. **Properties of the Resultant of a System of Forces.**—The resultant of a system of forces should be equivalent to the system in respect of every effect which a force can produce.

The resultant of a system of forces acting on a body of given mass should produce the same acceleration as the system itself. It will be seen that the connection between the parallelogram of forces and the parallelogram of accelerations depends upon this property of the resultant.

If a body acted on by a system of forces is held in equilibrium by the action on it of another force which exactly balances the system, the equilibrium should not be disturbed by substituting the resultant of the system of forces for the system itself. That is, the resultant of a system of forces should be equal in magnitude to the **equilibrant** of the system, and the two forces should act at the same point in opposite directions along the same line. In experimental work the resultant of a system of forces is usually found as in Exp. 5, by finding the equilibrant of the system and then reversing it.

The moment of the resultant of a system of forces round any point should be equal to the resultant or equivalent moment of the system round the same point. If the forces of the system are all in the same plane * or *co-planar forces*, the moment of the

* This is the only case that can be considered in elementary work. The composition of moments of force in different planes about a given point presents difficulties which cannot here be considered.

resultant of the system round any point in the plane is the algebraic sum of the moments of the individual forces of the system round the same point.

63. **Composition of Parallel Forces.**—Forces whose lines of action are parallel are called **parallel forces.** When two parallel forces act in the same direction they are said to be *like forces*, and when they act in opposite directions they are said to be *unlike forces.*

The resultant of two like parallel forces must evidently be equal to their sum, and must act in the same direction as the forces. The position of its line of action is determined by the fact that the moment of the resultant round any point is equal to the algebraic sum of the moments of the forces about the same point.

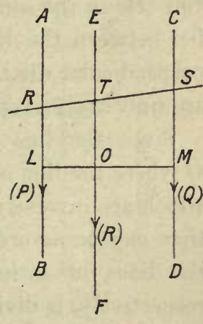

Fig. 63.

Hence, if a point O be supposed to be on the line of action of the resultant, the moment of the resultant round this point will be zero, and the algebraic sum of the moments of the forces round the point will also be zero. That is, the moments of the forces round the point O must be equal in magnitude and opposite in direction. The point O must, therefore, be between the two forces, and the position of the line of action of the resultant which passes through this point can be determined from the relation just stated.

Thus, let **two like parallel forces** of magnitude P and Q act along the parallel lines AB and CD, as shown in Fig. 63. The resultant of these two forces is of magnitude R, equal to (P + Q), and acts in the same direction as the forces. Its line of action, EF, lies between the lines AB and CD, and if O be a point on this line of action, the moment of P round O will be equal and opposite to the moment of Q round O. Hence, if LM be drawn through O at right angles to the direction of the forces, and cutting the lines AB and CD at L and M respectively, we have

$$P . OL = Q . OM,$$

or

$$\frac{OM}{OL} = \frac{P}{Q}.$$

That is, the line LM is divided at O into two segments, OL and OM, which are inversely proportional to the forces P and Q, to which these segments are adjacent. This fixes the position of the line of action of the resultant, for O is supposed to be a point on that line.

It follows, therefore, that the resultant of the two-like parallel forces of magnitude P and Q is of magnitude (P + Q), and acts in the same direction as the forces along a line which lies between the lines of action of the forces, and divides the perpendicular distance between them into two segments, which are universely proportional to the adjacent forces.

Since the line LM, in Fig. 63, is divided at the point O where the line of action of the resultant cuts it into segments which are inversely proportional to the adjacent forces, it follows that *any* transverse line RTS, Fig. 63, cutting AB and CD, the lines of action of the forces P and Q, at R and S respectively, is divided at the point T, where the line of action of the resultant cuts it, into two segments TR and TS, which are inversely proportional to the adjacent forces of magnitude P and Q.

Hence, if two like parallel forces P and Q act at the points A and B, Fig. 64, the line of action of their resultant cuts the line AB at the point C such that

$$\frac{CB}{CA} = \frac{P}{Q}.$$

Further, if we suppose the forces P and Q acting at A and B to change direction in any way, but always remaining like parallel forces, the line of action of the resultant will always divide AB at C in the ratio P : Q, and will, therefore, always pass through the same point C. That is, the resultant of two forces of magnitude P and Q, acting at the points A and B, will pass through the point C for all directions of the forces. The resultant of the two forces acting at the points A and B may, therefore, be supposed to act at the point C.

The resultant of **two unlike parallel forces** is equal to their difference, and acts in the direction of the greater of the two forces.

Also, if O be a point on the line of action of the resultant, the moment of the forces round this point must be equal in

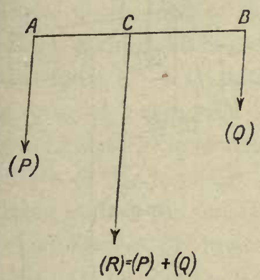

Fig. 64.

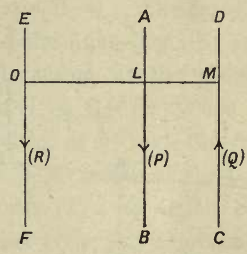

Fig. 65.

magnitude and opposite in direction, and the point must, therefore, have the lines of action of both forces on the same side of it, and be nearer to the greater than to the smaller force. That is, the line of action of the resultant must lie outside the lines of action of the forces on the side of the greater force.

Thus, let two unlike parallel forces of magnitude P and Q act along the parallel line AB and CD, as shown in Fig. 65. The resultant of these two forces is of magnitude R, is equal to (P − Q) if we assume P to be the greater, and acts in the same direction as P, the greater of the two forces. Its line of action, EF, lies outside the lines of action of the forces on the same side as the greater force P, and if O be a point on the line of action, we have, as in the case of like forces,

$$P \,.\, OL = Q \,.\, OM.$$

or

$$\frac{OM}{OL} = \frac{P}{Q}.$$

That is, the line LM is divided *externally* at O into two segments which are inversely proportional to the forces to which they are adjacent.

Hence, if two unlike parallel forces P and Q act at the points A and B, Fig. 66, the line of action of the resultant cuts

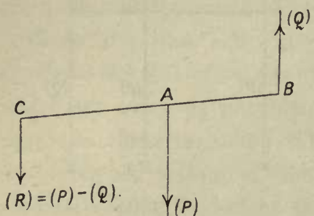

Fig. 66.

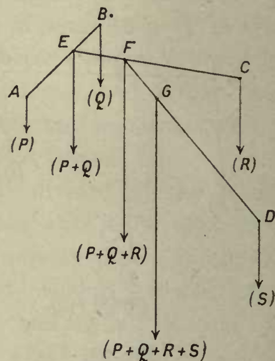

Fig. 67.

the line AB externally at a point C, such that

$$\frac{CB}{CA} = \frac{P}{Q}.$$

As explained above the resultant may be supposed to act at the point C, so that the resultant of the two unlike parallel forces P and Q acting at the point A and B, is a force of magnitude (P − Q) acting at the point C in the direction of the greater force.

It should be noticed that in Fig. 64 for like forces, and in Fig. 66 for unlike forces, the point C is so placed on the line

AB, or AB produced, that the distances from it of the points A and B, at which the forces P and Q act, are inversely proportional to these forces. It may be noticed, too, that in both figures, and in all similar figures, the greatest of the three forces P, Q, and R is in the middle position.

The resultant of any number of like parallel forces acting at given points, may be found by finding first the resultant of any two forces, then the resultant of the resultant so obtained and a third force, and so on until the final resultant of the complete system is obtained.

Thus, if four like parallel forces of magnitude P, Q, R, and S act at points A, B, C, and D, as shown in Fig. 67, the resultant of P and Q is a force of magnitude $(P + Q)$ acting at the point E, which divides AB inversely in the ratio $P : Q$. The resultant of this force $(P + Q)$ acting at E, and the force R acting at C, is a force of magnitude $(P + Q + R)$ acting at the point F which divides EC inversely in the ratio $(P + Q) : R$. Then the resultant of this force $(P + Q + R)$ acting at F, and the force S acting at D is the force $(P + Q + R + S)$ acting at the point G, which divides FD inversely in the ratio $(P + Q + R) : S$.

This final resultant acting at the point G is the resultant of the system of four like parallel forces of magnitudes P, Q, R, and S, acting at the points A, B, C, and D.

The point G is the point through which the resultant of the system would pass for all directions of the system, and is called the **centre of the system.** The centre of a system of parallel forces is thus a point whose position depends only on the magnitudes of the forces and the position of the points at which they act, and is quite independent of the direction of the system.

The resultant of any system of parallel forces may obviously be found by finding first the resultant of all the forces acting in one direction, then the resultant of all the forces acting in opposite direction, and finally the resultant of the two unlike forces thus obtained.

If, however, the forces are all in the same plane, the resultant is most readily found by making use of the facts that the magnitude of the resultant is the algebraic sum of the magnitudes of the forces of the system, and that the moment of the resultant round *any* point is the algebraic sum of the moments of the forces round the same point.

Most problems relating to parallel forces in one plane can be solved by making use of these two facts.

Examples.—1. A light rod, AB, 40 cms. long, carries a mass of 100 grammes at each end, and a mass of 300 grammes at a point 10 cms. from the end A; find a point on the rod such that if the rod is suspended by a string attached at this point it will hang in equilibrium in a horizontal position. The weight of the rod may be neglected.

The point required on the rod is the point at which the resultant of the weights of the suspended masses acts, for if the rod is suspended by a string attached at this point the resultant of the weights and the tension on the string will act in the same vertical line, and the rod will be in equilibrium in any position.

The magnitude of the resultant of the weight is (100 + 300 + 100) gramme-weights, or 500 gramme-weights, and if the distance of the point at which it acts from the end A of the rod be denoted by x, we have, by taking moments round A when the rod is horizontal,

$$500x = 300 \times 10 + 100 \times 40;$$

or

$$500x = 7,000,$$

and

$$x = 14.$$

That is, the resultant of the weights acts at a point on the rod 14 cms. from the end A, and if the rod is suspended by a string attached at this point it will hang in equilibrium in a horizontal position, or in a position making any angle with the horizontal.

2. A rod 4 feet long is suspended in a horizontal position by two vertical strings attached at the ends of the rod, and masses of 3 lbs., 4 lbs., and 5 lbs. are suspended from the rod at distances of 1 ft., 2 ft., and 3 ft. from one end. Find the tension in each of the strings by which the rod is suspended, the weight of the rod itself being neglected.

Let the arrangement of the rod and the masses suspended from it be as shown in Fig. 68, and let T_1 and T_2 denote, in pound-weights, the tensions in the strings at the ends A and B of the rod AB.

Since the rod is in equilibrium the algebraic sum of the moments of the forces round any point in their plane must be zero.

Hence, if we take moments round A we must have

$$4\,T_2 - 3 \times 1 - 4 \times 2 + 5 \times 3 = 0,$$

or $$4\,T_2 = 3 + 8 + 15 = 26,$$

and $$T_2 = 6\tfrac{1}{2}.$$

That is, the tension in the string supporting the rod at the end B is $6\frac{1}{2}$ pound-weights. Similarly, if we take moments round B, we get

$$4\,T_1 = 22,$$

or $$T_1 = 5\tfrac{1}{2}.$$

That is, the tension in the string supporting the end B of the rod is $5\frac{1}{2}$ pound-weights.

It should be noticed that by taking moments round the point A the

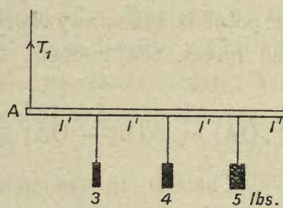

Fig. 68.

unknown force, T_1, *acting at that point*, is eliminated from the equation obtained. Similarly, by taking moments round the point B we eliminate T_2. If moments are taken round any other point the equation of moments must contain T_1 and T_2, and a second equation derived from the fact that the resultant of the system is of zero magnitude must be used to obtain the values of T_1 and T_2.

This second equation, $T_1 + T_2 - 3 - 4 - 5 = 0$, or $T_1 + T_2 = 12$, may be used in the solution given above after finding the value of T_2. For if we substitute the value $6\frac{1}{2}$ obtained for T_2 in this equation, we get

$$T_1 + 6\tfrac{1}{2} = 12,$$

or $$T_1 = 5\tfrac{1}{2}.$$

64. **A Couple.**—When two unlike parallel forces are equal in magnitude they have no real resultant, and are said to form a **couple.** The effect of the action of a couple on any body

is to produce rotation, and the proper measure of a couple is its moment.

The moment of a couple round any point is the algebraic sum of the moments of the two forces which constitute the couple round the same point, and it can be shown that it has the same value for every point in the plane of the couple. Thus, let the two unlike parallel forces of equal magnitude P constitute a couple, and consider the moment of the couple round any point O, Fig. 69, in the plane of the couple. When the point O is taken anywhere between the lines of action of the two forces of the couple the moment of the couple is given by

$$(P \,.\, OA + P \,.\, OB) = P(OA + OB) = P \,.\, AB.$$

Similarly, when the point is taken anywhere external to the lines of action of the forces, the moment of the couple is given by

$$(P \,.\, OB - P \,.\, OA) = P(OB - OA) = P \,.\, AB.$$

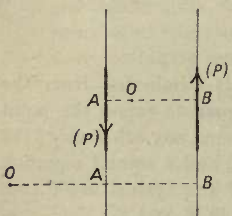

Fig. 69.

That is, the moment of the couple round *any point* in its plane is given by (P . AB), where AB is the perpendicular distance between the lines of action of the forces of the couple. The moment of the couple round any point in its plane is, therefore, of constant value, for the distance AB is obviously constant. This distance is called the *arm* of the couple, and the *moment of a couple* is measured by the product of the magnitude of either force of the couple into the arm of the couple.

It will be seen that two couples of equal moment in the same plane must balance each other if their moments are of opposite sign, for, wherever they act in the plane the algebraic sum of their moments round any point in the plane must be zero.

It should be noted that whenever a body moves with motion of rotation only it is subject to the action of a couple. In many cases, however, only one force is applied directly, the second force of the couple being supplied by the reaction at the axis of rotation. For example, when a door or gate hung on hinges is pushed open it rotates on its hinges round a vertical axis. If a force is applied to the door in a horizontal plane at right angles to the plane of the door the resultant reaction at the hinges is equal and opposite to it and acts at a point on the axis of rotation in the same horizontal plane as the applied force. The applied force and the reaction at the axis of rotation thus constitute the couple which causes the rotation of the door on its hinges.

65. **Composition of a System of Co-planar Forces.**—A system of forces whose lines of action all lie in the same plane is called a system of **co-planar forces.** The forces of the system are generally supposed to act on the same rigid body at different points in the plane.

A system of this kind, if not in equilibrium, can evidently be reduced by compounding the component forces, to a single resultant force or to a couple. For, if the forces are more than two in number it is always possible to compound two of them into a single resultant force, and so reduce the number of forces by one. When the forces are reduced in this way to two in number these two forces must either be capable of being compounded into a single resultant force, or they must be equal in magnitude and act in opposite directions along the *same straight line,* and so be in equilibrium, or along *parallel straight lines,* and so form a couple.

66. **Resolution of Forces.**—A force being a vector quantity may be resolved into two components either by the parallelogram rule or the triangle rule. Thus, if the line AC in Fig. 70 represent a force in magnitude and direction, and if we construct on AC as diagonal *any* parallelogram such as

ABCD, then the two adjacent sides AB and AD represent the two component forces into which the force represented by AC is in this way resolved. Similarly, if we construct on AC any triangle, such as ABC, then the two sides AB and BC represent the two component forces into which the force represented by AC is thereby resolved.

In the same way by applying the polygon of forces any given force may be resolved into any number of co-planar components acting at the same point. Thus, if AF, in Fig. 71, represent the given force in magnitude and direction, then, by constructing on AF any polygon, such as ABCDEF, the given force is resolved into five components represented in magnitude

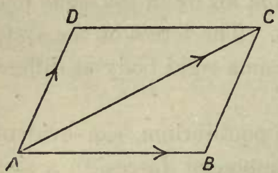

Fig. 70.

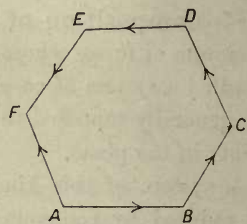

Fig. 71.

and direction by the lines AB, BC, CD, DE, and EF. That is, if the five forces represented by these lines act at a point, they will have the given force represented by AF as their resultant.

The most important case of resolution of a force is that in which the force is resolved into two components at right angles. The usual method of resolving a vector quantity into two components at right angles has already been dealt with in Art. 9.

It will be seen that if a force of magnitude **F**, acting along OA, as shown in Fig. 72, be resolved along any line OX, making an angle θ with OA, and along OY at right angles to

OX, the two components are **F cos** θ along OX, and **F sin** θ along OY.

This result may be expressed in other terms, but it will be found that the form here adopted is, in the end, the simplest and the most generally useful.

The resultant of a number of co-planar forces acting at a point can sometimes be found most conveniently by resolving each force into two components along any two directions taken at right angles to each other through the point, and then finding the resultant of the algebraic sums of the components along the two directions of resolution. Thus, if the forces acting at the point O, in Fig. 73, be resolved individually along the two

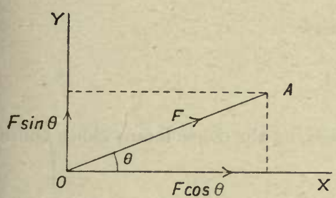

Fig. 72.

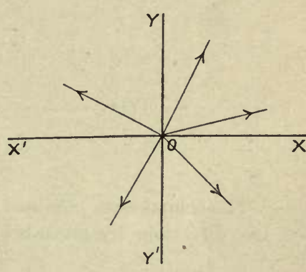

Fig. 73.

rectangular axes X′OX and Y′OY, and if X denote the algebraic sum of the components along X′OX, and Y denote the algebraic sum of the components along Y′OY, then R, the resultant of the system of forces acting at O, is given by

$$R^2 = X^2 + Y^2.$$

Example.—Five forces, P, Q, R, S, T, of magnitude 15 units, 6 units, 10 units, 12 units, and 20 units respectively, act at a point in a vertical plane. The force P acts horizontally to the right, and the lines of action of the other forces, taken in the order given, make angles of 30°, 90°, 120°, and 225° (measured counter-clockwise) with the line of action of this force. Find the resultant of the system.

Let the forces of the system act at the point O as represented in Fig. 74.

Resolve the forces along two directions, X′OX and Y′OY, taken through O at right angles to each other, and coinciding with the directions of the forces P and R. The forces P and R already act along these directions, and need not, therefore, be resolved.

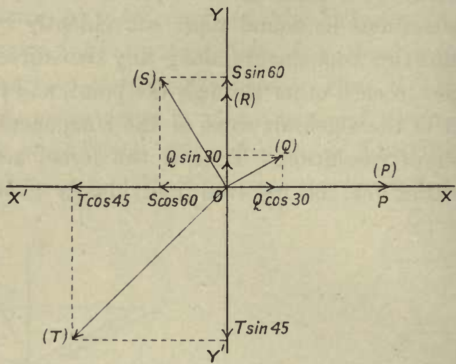

Fig. 74.

The components obtained by resolving the other forces along these two directions are given below.

Forces.	Components along	
	X′OX.	Y′OY.
P (15 units)	+ 15	0
Q (6 units)	$+ 3\sqrt{3}$ units	+ 3 units
R (10 units)	0	+ 10
S (12 units)	− 6 ,,	$+ 6\sqrt{3}$ units
T (20 units)	$- 10\sqrt{2}$,,	$- 10\sqrt{2}$,,

The algebraic sum of the components along X′OX is, therefore, $(15 + 3\sqrt{3} - 6 - 10\sqrt{2})$ units, or 3·52 units; and the algebraic sum along Y′OY is $(10 + 3 + 6\sqrt{3} - 10\sqrt{2})$ units, or 9·25 units.

The system is thus reduced to a force of 3·52 units acting along OX, and a force of 9·25 units acting along OY. The magnitude of

R, the resultant of the system, is given, therefore, by

$$R^2 = (3{\cdot}52)^2 + (9{\cdot}25)^2,$$

or, $$R = 9{\cdot}9 \text{ nearly}.$$

That is, the resultant is a force of nearly 9·9 units acting at O in a direction which makes with OX an angle whose tangent is $\frac{925}{352}$, or 2·63. The magnitude of this angle is about 67° 40′.

The resultant of the given system of forces is thus completely determined in magnitude and direction.

CHAPTER XII.

CENTRE OF GRAVITY.

67. **Centre of Gravity.**—If a force is applied to a block of stone by means of a rope attached to the block at a particular point, the force can be said to be applied to the block at the point of attachment of the rope.

The force of attraction exerted by the earth on any body near it cannot, however, be said to be applied to the body at any particular point. The weight of each particle of the body acts on the particle at the point where it is situated in the body, and the weight of the body as a whole is really a system of parallel forces, infinite in number, made up of the weights of all the particles which make up the body, acting at the points where these particles are situated.

If a body held in any position is rotated through any angle into another position the direction of the system of parallel forces which constitutes the weight of the body does not change *relative to the earth*, for it is always vertical, but it does change *relative to a fixed line in the body*. If the body, for example, is rotated through any angle in a vertical plane, the direction of the system in the body relative to a fixed line in the body changes through the same angle. That is, as the position of the body is changed, the forces of the system remain of the same magnitude and act at the same points *in the body*, but the direction of the system relative to any fixed line *in the body*, in general, changes. It follows, therefore, as explained in Art. 63, that the resultant of the system passes, for all positions of the body, through a point which is fixed in position relative to the body. If the system is considered relative to the body only, this point is the centre of the system, as defined in Art. 63, and is called the **centre of gravity** of the body.

The centre of gravity of a body may, therefore, be defined as the point through which the resultant of the weights of the particles which make up the body passes for all positions of the body. The weight of the body may thus be considered as a single resultant force, acting at the centre of gravity of the body.

It will be understood that the centre of gravity of a body is not necessarily in the body itself; it may be at a point outside the material of the body, but the position of the point is fixed relative to the body and must be specified in this relation. For example, the centre of gravity of a length of wire bent in the form of a circular ring is at the centre of the ring.

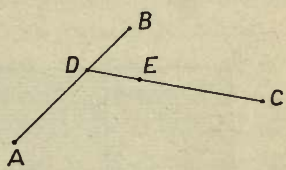

Fig. 75.

When we speak of the centre of gravity of a body as a definite fixed point in the configuration of the body it is understood that the body is rigid. If the body is made up of movable parts the centre of gravity is fixed for any given configuration of the body, but changes its position with change of configuration. For example, the position of the centre of gravity of a bicycle depends upon the arrangement of its parts, but for any given arrangement it is a definite fixed point.

68. **Method of Finding the Position of the Centre of Gravity of a Body.**—The method of finding theoretically the centre of gravity of a body is based on the result given in Art. 63. Thus, if two particles of weights ω_1 and ω_2 are placed at points A and B, Fig. 75, their centre of gravity is on the line AB at the point D which divides the line into two segments AD and DB is the ratio $\omega_2 : \omega_1$. Similarly,

the centre of gravity of three particles of weights ω_1, ω_2, and ω_3 at the points A, B, and C is on the line CD at the point E which divides the line into two segments DE and EC in the ratio $(\omega_1 + \omega_2) : \omega_3$.

Any body may be supposed to be made up of its constituent particles, and as we can, by the method here indicated, find the centre of gravity of any system of particles, we can, in theory, find the centre of gravity of any body.

In simple cases where the body is of regular form, and its particles are arranged symmetrically about any point or line in the body the application of the method is comparatively easy.

Thus, the centre of gravity of a straight uniform filament, such as a straight piece of very fine uniform wire, is evidently at

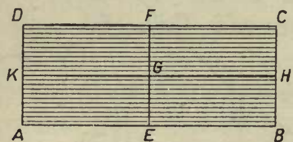

Fig. 76.

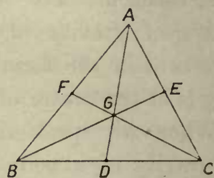

Fig. 77.

its middle point. The filament may be considered as a row of particles of equal weight distributed uniformly along a straight line, and by pairing particles equidistant from the centre it can be seen that the centre of gravity of every pair, and, therefore, the centre of gravity of the filament, as a whole, is at the middle point.

From this result the centre of gravity of any regular plane *lamina*, or thin sheet of matter uniformly distributed over any regular plane area, can readily be found. Thus, the rectangular lamina ABCD in Fig. 76 may be supposed to be made up, as indicated in the figure, of an infinite number of straight filaments or *linear elements* arranged parallel to the sides AB and CD. The centre of gravity of each of these elements is at its middle point, so that the centres of gravity of all the elements which make up the lamina lie along the *median line* EF which joins the mid points, E and F, of the lines AB and CD. It follows from this that the centre of gravity of the lamina must be somewhere on this median line.

In the same way, by dividing the lamina into linear elements parallel to the sides AD and BC, it can be shown that the centre of gravity must lie on the median line HK. Hence, if the centre of gravity of the lamina lies on the line EF and also on the line HK, it must be at the point G where these two lines intersect.

In this case it might have been stated at once, after showing that the centre of gravity of the lamina lies on the median line EF, that it lies at the middle point of the line. The linear elements into which the lamina is divided parallel to AB and CD are equal, and their weights acting at their centres of gravity on the median line EF are uniformly distributed along this line. It follows, therefore, that the centre of gravity of the lamina made up of these elements must be at the middle of the line.

It will be seen from what has been said above that the centre of gravity of a lamina in the form of a square, a rectangle, or a parallelogram, is at the intersection of the median lines of the figure. It is easily shown geometrically that this point is also the intersection of the diagonals of the figure.

In the same way it can be shown that the centre of gravity of a lamina in the form of any regular plane figure, is at the intersection of any two median lines of the figure. Thus, the centre of gravity of a circular lamina is at its centre, an the centre of gravity of a lamina in the form of any regular polygon is at the centre of the circle circumscribing the polygon.

In the case of a triangular lamina ABC, Fig. 77, it can be shown by dividing the lamina into linear elements parallel to any one of the sides, that the centre of gravity must lie on the median line joining the middle point of that side to the opposite angular point, and must, therefore, be at G, the point of intersection of any two of the three median lines AD, BE, and CF. It is easily proved geometrically that this point is so placed on each of the median lines that its distance from the mid-point of the side to which the line is drawn is one-third the length of the line. That is, $DG = \frac{1}{3} DA$, $EG = \frac{1}{3} EB$, and $FG = \frac{1}{3} FC$.

Just as a lamina may be divided for the purpose of finding its centre of gravity into linear elements, and a linear element into particles, so a solid body may be divided into infinitely thin laminar elements or slices.

If the laminar elements into which a solid can be divided are of any regular form for which the centre of gravity is known, and if they are arranged in a regular and definite manner, the centre of gravity of the solid can, in general, be found by an extension of the method indicated above.

A cylinder, for example, may be divided into an infinite number of circular laminæ or slices at right angles to the axis; the centre of gravity of each lamina is at its centre, on the axis of the cylinder, and as the laminæ are distributed uniformly along the axis, the centre of gravity of the cylinder must be at the middle point of its axis. Each lamina may be supposed to be replaced by a particle of the same weight placed at the centre of gravity of the lamina on the axis of the cylinder. The cylinder as a whole thus becomes equivalent to a line of *equal* particles *uniformly* distributed along the axis, and its centre of gravity must, therefore, be at the middle point of the axis.

Similarly, the centre of gravity of any prism whose cross section is a regular polygon, is at the middle point of its axis, and the centre of gravity of a triangular prism is at the middle point of the line parallel to its edges, which passes through the centres of gravity of the transverse triangular laminæ, into which the prism may be divided.

The centre of gravity of a sphere is obviously, from considerations of symmetry, at the centre of the sphere. It can be seen, also, that if the sphere is divided into laminar elements at right angles to any diameter, the centre of gravity must lie on that diameter, and must, therefore, be at the point where any two diameters intersect. This point is the centre of the sphere.

The centre of gravity of a cone must evidently lie on the line joining the apex of the cone to the centre of the base, for if the cone be supposed divided into laminar elements parallel to its base, the centre of gravity of each element lies on this line, and the centre of gravity of the cone made up of these elements must, therefore, be at a point on the line.

The elements are not, however, of equal weight, so that the weight is not distributed uniformly along this line, and the centre of gravity is, consequently, not at the middle point of the line, but at a point nearer the base of the cone. The exact position of the centre of gravity of the cone on this line evidently depends upon the manner in which the weight is distributed along this line. It can be seen without much difficulty that the weight at any point on the line is proportional to the square of the distance of the point from the apex of the cone, and it can be shown from this that the centre of gravity of the cone is at a point on the line whose distance from the base of the cone is one-fourth the length of the line. In the same way it is found that the centre of gravity of a pyramid is on the line joining the apex of the pyramid to the centre of gravity of the base at a point whose distance from the base is one-fourth the length of the line.

It will be seen from the examples given above that the usual method of finding the centre of gravity of a body is to divide the body into suitable elements for each of which the position of the centre of gravity is known. If the centres of gravity of these elements all lie on a straight line, the centre of gravity of the body must lie on that line, and if another similar straight line can be found, the centre of gravity must be at the point of intersection of the two straight lines. The exact position of this point can then be found by geometry.

If there is only one way of dividing the body into elements whose centres of gravity all lie on a straight line, the elements may be supposed to be replaced by particles of the same weight placed at their centres of gravity. The problem of finding the centre of gravity of the body is thus reduced to finding the centre of gravity of a line of particles of known weight arranged along a straight line.

The most general method of finding the centre of gravity of a number of particles whose weights and positions are known is indicated below for particles in one plane.

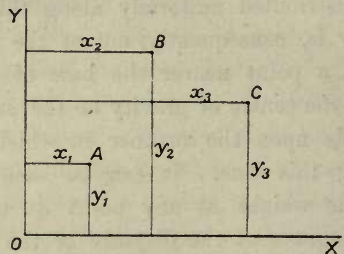

Fig. 78.

Let $\omega_1, \omega_2, \omega_3, \ldots \omega_n$ denote the weights of a number of particles at the point A, B, C, . . . N in the plane of the paper, Fig. 78, and let (x_1, y_1), (x_2, y_2), (x_3, y_3), . . . (x_n, y_n) be the co-ordinates of these points with reference to the axes OX and OY. Then, if $\bar{x}$ and $\bar{y}$ denote the co-ordinates of the centre of gravity of the particles, we have

$$\bar{x} = \frac{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 \quad . \quad . \quad . \quad + \omega_n x_n}{\omega_1 + \omega_2 + \omega_3 \quad . \quad . \quad . \quad + \omega_n},$$

and

$$\bar{y} = \frac{\omega_1 y_1 + \omega_2 y_2 + \omega_3 y_3 \quad . \quad . \quad . \quad + \omega_n y_n}{\omega_1 + \omega_2 + \omega_3 \quad . \quad . \quad . \quad + \omega_n}.$$

That is,

$$\bar{x} = \frac{\Sigma(\omega x)}{\Sigma(\omega)},$$

and

$$\bar{y} = \frac{\Sigma(\omega y)}{\Sigma(\omega)}.$$

This result is readily obtained geometrically by working out the co-ordinates of the centre of gravity of the particles by applying the method of Art. 63. Thus, the co-ordinate of the centre of gravity of the particles at A and B obtained by this method will be found to be $\left(\frac{\omega_1 x_1 + \omega_2 x_2}{\omega_1 + \omega_2}\right)$, and $\left(\frac{\omega_1 y_1 + \omega_2 y_2}{\omega_1 + \omega_2}\right)$, and continued application of this result leads for any number of particles to the general result given above.

The result is, however, more easily obtained by supposing the particles to be in a horizontal plane, so that their weights act at right angles to the plane of the axes of co-ordinates, and then taking moments about OX to find $\bar{x}$, and about OY to find $\bar{y}$.

The moment of the resultant of the weights of the particles, *acting at their centre of gravity*, about either axis, must be equal to the algebraic sum of the moments of the weights of the individual particles about the same axis. Hence, by taking moments about OX we have

$$(\omega_1 + \omega_2 + \omega_3 \ldots \omega_n)\bar{x} = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 \ldots + \omega_n x_n,$$

or

$$\bar{x} = \frac{\Sigma(\omega x)}{\Sigma(\omega)} \text{ as above.}$$

Similarly, by taking moments round OY we get

$$\bar{y} = \frac{\Sigma(\omega y)}{\Sigma(\omega)}.$$

In these results the expressions $\Sigma(\omega x)$ and $\Sigma(\omega y)$ must be taken to mean the *algebraic sum* of the quantities of the form ωx

and ωy for, unless the particles are all in one quadrant, as in Fig. 78, the co-ordinates of the points at which the particles are placed may be positive or negative.

If the particles are distributed along a straight line their centre of gravity is also on this line, and if the line be taken as the axis of x, then we have

$$\bar{x} = \frac{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \ldots + \omega_n x_n}{\omega_1 + \omega_2 + \omega_3 \ldots + \omega_n},$$

where $x_1, x_2, x_3, \ldots x_n$ are the distances of the particles from *any point on the line*, taken as origin, and $\bar{x}$ is the distance of the centre of gravity from the same point.

The determination of the centre of gravity of a body by dividing the body into infinitesimal elements, and then applying the results given above, usually requires advanced mathematical methods, and cannot be further considered. If, however, any body can be divided into a few finite parts for which the positions of the centres of gravity are known, the parts may be supposed to be replaced by particles of the same weight at their centres of gravity; and the centre of gravity of these particles—that is, the centre of gravity of the body—can then be found by the method of Art. 63, or by applying the results given above.

Examples.—1. Find the centre of gravity of a uniform lamina in the form of an irregular quadrilateral. The lamina ABCD, shown in Fig. 79, may be divided into two triangular luminæ, ABC and ADC. Draw the diagonal AC and let O be its middle point. The centre of gravity of the triangular lamina ABC will then be on the median line BO at a point E, such that OE = $\frac{1}{3}$ OB. Similarly, the centre of gravity of the triangular lamina ADC is on the median line DO at a point F, such that OF = $\frac{1}{3}$ OD.

The triangular laminæ ABC and ADC may therefore be supposed to be replaced by particles of the same weights, placed at the points E and F respectively. The centre of gravity of these two particles will be on the line EF at a point G which divides the line into two segments, which are inversely proportional to the weights of the adjacent particles. The weights of the particles at E and F are evidently proportional to the areas of the triangles ABC and ADC, and the line EF must therefore be divided at G, so that

GE : GF :: Area of triangle ADC : Area of triangle ABC.

This relation determines the position of the point G on the line EF, and this point is the centre of gravity of the lamina.

Instead of finding the position of G on the line EF in the manner given above, it would be possible, by dividing the lamina ABCD into two triangular laminæ BAD and BCD, to find another line, E′F′, on which the centre of gravity must lie. The point G would then be determined by the intersection of the two lines EF and E′F′.

2. Find the centre of gravity of a uniform wire bent into the form of a triangle.

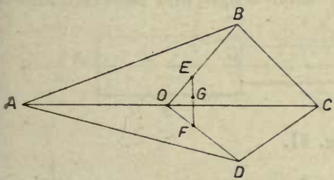

Fig. 79.

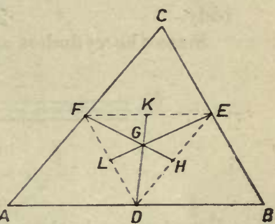

Fig. 80.

The bent wire ABC, Fig. 80, may be divided into three straight lengths AB, BC, and CA, and the centres of gravity of these lengths are at their mid-points D, E, and F.

The three lengths AB, BC, and CA may, therefore, be replaced by particles of the same weights at the points D, E, and F. The weights of the particles at D, E, and F are thus proportional to the lengths of the sides AB, BC, and CA of the triangle ABC.

The centre of gravity of the two particles at D and E will, therefore, be on the line DE at a point H, which divides the line, so that we have HD : HE :: BC : AB. The two particles at D and E may, therefore, be replaced by a single particle of their combined weight at the point H, and it follows that the centre of gravity of the three particles at D, E, and F, or the centre of gravity of the bent wire, must lie on the line FH. Similarly, it can be seen that the centre of gravity of the two particles at E and F is on the line EF at a point K, such that

$$EK : KF :: CA : BC,$$

and it follows that the centre of gravity of the bent wire must lie on the line DK.

Since the centre of gravity of the wire is thus found to lie on the line FH, and also on the line DK, it must be at the point G where these two lines intersect.

By following up the geometry of the figure with the help of *Euc.* vi. 3, it can be seen that the lines FH and DK bisect respectively the angles EFD and FDE ; and that the point G, the centre of gravity of the bent wire, is the centre of the circle inscribed in the triangle DEF.

3. A body of given material is made in the form of three cylinders arranged end to end with their axes in the same line. The cylinders are of the same length, but their diameters in order are in the ratio 1 : 2 : 3. Find the position of the centre of the gravity of the body.

Since the cylinders are of the same length and material, and their

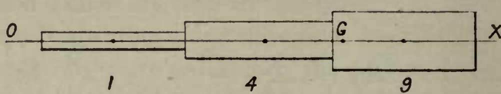

Fig. 81.

diameters are in the ratio 1 : 2 : 3, their weights must be in the ratio 1 : 4 : 9.

Hence, if $2l$ denote the length of each cylinder, and distances be measured along the common axis of the cylinders from the narrow end of the body as origin, we get, by dividing the body into the three cylindrical parts of which it consists, and then applying the relation,

$$\bar{x} = \frac{\Sigma(\omega x)}{\Sigma(\omega)}.$$

That is,
$$\bar{x} = \frac{(1 \times l) + (4 \times 3l) + (9 \times 5l)}{1 + 4 + 9},$$

or
$$x = \frac{58l}{14} = \frac{29l}{7}.$$

That is, the centre of gravity of the body is at a point G, Fig. 81, such that $OG = 4\frac{1}{7}\,.\,l$, where l is half the length of each cylinder.

4. A circular lamina has a circular piece cut out ; the radius of the piece cut out is one-half the radius of the lamina, and its centre is at a point whose distance from the centre of the lamina is one-fourth of

the radius of the lamina. Find the centre of gravity of the portion of the circular lamina which remains after the circular piece is cut out.

The centre of gravity of the lamina as a whole is at its centre G, Fig. 82.

The centre of gravity of the piece cut out is at a point A whose distance from G is one-fourth the radius of the lamina.

The centre of gravity of the remaining portion of the lamina must, therefore, be at some point X on the line AG produced. This is evident, for the centre of gravity of the whole lamina must lie on the line joining the centres of gravity of the two parts here considered. That is, the points AG and X must lie in a straight line.

If the weight of the piece cut out be taken as 1 unit, the weight of the whole lamina will be 4 units, and the weight of the portion remaining will be 3 units.

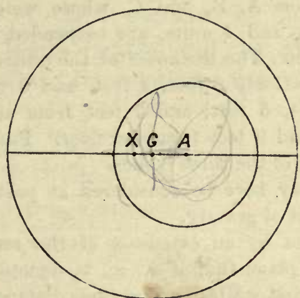

Fig. 82.

Hence, if the distance GX be denoted by x and the radius of the lamina by r, and we apply the relation,

$$\bar{x} = \frac{\Sigma(wx)}{\Sigma(w)}, \text{ we get,*}$$

$$0 = \frac{\left(1 \times \frac{r}{4}\right) - 3x}{4},$$

or $$3x = \frac{r}{4},$$

and $$x = \frac{r}{12}.$$

* It should be noticed that, in this relation, when distances are measured from the centre of gravity of the body as origin, we must have $\bar{x} = 0$. In this case with G as origin the distance GA is positive and GX negative according to the usual convention.

That is, the centre of gravity of the portion of the lamina which remains after the piece is cut out, is on the line AG produced, at a point X whose distance from G, the centre of the lamina, is one-twelfth of the radius of the lamina.

Instead of applying the general relation, as above, it would be simpler in this case simply to take moments about G. If the line AGX s supposed to be horizontal the weights of the lamina and the two parts considered act vertically at right angles to AGX, and by taking moments round G, we get

$$3x = 1\frac{r}{4},$$

or

$$x = \frac{r}{12}.$$

5. Three bodies A, B, and C, whose weights are respectively 2 units, 3 units, and 4 units, are suspended by threads from the ceiling of a room. The distances of the centres of gravity of these bodies are respectively 2 feet, 4 feet, and 6 feet from one wall of the room, 3 feet, 5 feet, and 7 feet from an adjacent wall, and 3 feet, 6 feet, and 9 feet from the floor. Find the position of the centre of gravity of the three bodies.

The bodies may here be considered as particles placed at their respective centres of gravity.

It will be seen by an extension of the result given above for particles in one plane that if x_1, x_2, x_3 denote the distances of the bodies from the first wall, y_1, y_2, y_3 their distances from the adjacent wall, and z_1, z_2, z_3 their distances from the floor, we have

$$\bar{x} = \frac{\Sigma(\omega x)}{\Sigma(\omega)}, \quad \bar{y} = \frac{\Sigma(\omega y)}{\Sigma(\omega)}, \quad \bar{z} = \frac{\Sigma(\omega z)}{\Sigma(\omega)}$$

where x, $\bar{y}$, and $\bar{z}$ denote the distances of the centre of gravity of the three bodies from the reference planes specified above.

Hence, by applying these results we get

$$\bar{x} = \frac{(2 \times 2) + (3 \times 4) + (4 \times 6)}{2 + 3 + 4} = \frac{40}{9} = 4\tfrac{4}{9},$$

$$\bar{y} = \frac{(2 \times 3) + (3 \times 5) + (4 \times 7)}{2 + 3 + 4} = \frac{49}{9} = 5\tfrac{4}{9},$$

$$\bar{z} = \frac{(2 \times 3) + (3 \times 6) + (4 \times 9)}{2 + 3 + 4} = \frac{60}{9} = 6\tfrac{2}{3}.$$

That is, the centre of gravity of the three bodies is at a point distant $4\frac{4}{9}$ feet from the first wall, $5\frac{4}{9}$ feet from the adjacent wall, and $6\frac{2}{3}$ feet from the floor.

69. **Centre of Mass.**—It has been explained in Art. 68 that if particles of weights $\omega_1, \omega_2, \omega_3 \quad . \quad . \quad . \quad \omega_n$ lie in the same plane at points whose co-ordinates with reference to known axes are $(x_1 y_1)$, $(x_2 . y_2)$, $(x_3 . y_3) \quad . \quad . \quad . \quad (x_n . y_n)$, the position of the centre of gravity of the particles is given by the relations

$$\bar{x} = \frac{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 \quad . \quad . \quad . \quad + \omega_n x_n}{\omega_1 + \omega_2 + \omega_3 \quad . \quad . \quad . \quad + \omega_n},$$

and

$$\bar{y} = \frac{\omega_1 y_1 + \omega_2 y_2 + \omega_3 y_3 \quad . \quad . \quad . \quad + \omega_n y_n}{\omega_1 + \omega_2 + \omega_3 \quad . \quad . \quad . \quad + \omega_n}.$$

If the masses of these particles be denoted by m_1, m_2, $m_3 \quad . \quad . \quad . \quad m_n$ their weights will be given by $m_1 g$, $m_2 g$, $m_3 g \quad . \quad . \quad . \quad m_n g$, and the relations given above reduce to

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 \quad . \quad . \quad . \quad + m_n x_n}{m_1 + m_2 + m_3 \quad . \quad . \quad . \quad + m_n},$$

and

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 \quad . \quad . \quad . \quad + m_n y_n}{m_1 + m_2 + m_3 \quad . \quad . \quad . \quad + m_n}.$$

It will be seen that in making the necessary substitutions the quantity g appears in both numerator and denominator of these relations, and, therefore, cancels out in the final form given above.

It follows from this result that the position of the point whose co-ordinates are $(\bar{x}, \bar{y})$, really depends upon the distribution of mass in the plane. That is, the position of the point called the centre of gravity of the body really depends upon the distribution of mass in the body, and is really the *centre of mass* of the body.

The centre of mass of a body is most conveniently defined by the method indicated in Example 5 in Art. 68. If the distance of any particle in the body from three reference planes at right angles to each other and having a common point at the origin be denoted by x, y, and z, and if the mass of the particles be denoted by m, then the point whose co-ordinates are

$$\bar{x} = \frac{\Sigma(mx)}{\Sigma(m)}, \quad \bar{y} = \frac{\Sigma(my)}{\Sigma(m)}, \quad \bar{z} = \frac{\Sigma(mz)}{\Sigma(m)}$$

is the centre of mass of the body.

It will be seen that if a body is subject to the action of any force, such that the forces to which the individual particles of the body are subject are all parallel in direction, and *proportional to the masses of the particles* in magnitude, the centre of this system of parallel forces, and the point at which the force acting on the body may be supposed to act, is the centre of mass of the body.

The attraction of the earth on any body is a force of this kind, as explained in Art. 67, and it follows that the centre of gravity of a body is at its centre of mass.

The centre of mass of a body has some important dynamical properties. When a body is moving with pure motion of translation its velocity, as explained in Art. 26, is the velocity of any point in it; but, if the body is moving in any way, its velocity, in accordance with Newton's laws of motion, is the velocity of its centre of mass. Similarly, the momentum of a body moving in any way is given by the product of the mass of the body into the velocity of its centre of mass.

It can be shown, too, that the kinetic energy of a body in motion is the kinetic energy of the whole mass moving with the velocity of the centre of mass, together with the sum of the kinetic energies of its component particles relative to the centre of mass.

70. **Experimental Method of Finding the Centre of Gravity of a Body.**—If a body is suspended by a thread attached to the body at any point, the body will hang when at rest with its centre of gravity vertically below the point of suspension. The forces acting on the suspended body are its

weight acting vertically downwards at its centre of gravity G, as shown in Fig. 83, and the tension in the suspension thread acting vertically upwards at the point of suspension A. When the body hangs at rest these two forces balance each other; they must therefore be equal in magnitude, and must act in opposite directions along the same straight line. That is, the vertical through G and the vertical through A are in the same straight line, and the point G is therefore vertically below the point A.

Hence, if a body is suspended by a thread attached at any point A, the centre of gravity of the body lies on the prolongation of the direction of the thread through A. Similiarly, if the body is suspended by a thread attached at another point B, the centre of gravity lies on the prolongation of the direction of the thread through B. The centre of gravity will therefore be found at the intersection of the two directions thus determined in the body.

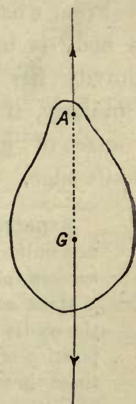

Fig. 83.

In many cases it is practically impossible to apply this method experimentally for the directions of the lines on which the centre of gravity is known to lie cannot in general be marked, and the point at which they intersect cannot be determined. In the case of some bodies, however, of open structure, such as a bird cage or any open frame work, it is generally possible to mark the prolongations of the suspension thread by means of threads, and so to fix the position of the centre of gravity at the point where these threads intersect. In the case of thin plane sheets or laminæ of any rigid material, the centre of gravity is easily found by this method as explained below.

Experiment 6.—Take a plane sheet of cardboard or tin-plate of any irregular shape, and find the position of its centre of gravity by suspending it by a thread attached successively at two different points, A and B, taken anywhere at the edge of the sheet. When the sheet hangs at rest by a thread attached at the point A, mark the prolongation of the direction of the thread on the face of the suspended sheet by means of a plumb line through A. Similarly, when the sheet hangs at rest by a thread attached at the point B, mark the prolongation of thread in the same way on the face of the sheet.

The point at which the two lines thus drawn on the face of the sheet intersect is the centre of gravity of the sheet. Suspend the sheet by attaching the thread at other points, and note that the prolongation of the direction of the thread in every case passes through the same point.

From what has been said above, it will be understood that if a body is balanced at rest on a point or pivot, the centre of gravity lies on the vertical line passing through the point. Similarly, if a body is balanced at rest on a knife-edge the centre of gravity lies in the vertical plane through the knife-edge.

Experiment 7.—Find the centre of gravity of a rectangular block of non-uniform material by balancing it on a knife-edge taken parallel successively to the length, breadth, and thickness of the block.

Mark on the block in each case the position of the plane in which the centre of gravity lies, and specify from the data thus obtained the position of the centre of gravity as determined by the point at which these three planes intersect.

CHAPTER XIII.

EQUILIBRIUM OF FORCES.

71. **Equilibrium.**—When the forces acting on a body balance each other so that they cannot produce any change in the body's state of rest or motion the forces are in **equilibrium.**

It follows from this that the magnitude of the resultant of a system of forces in equilibrium must be zero, and that the algebraic sum of the moments of the forces about any and every point must be zero.

If the resultant is zero there can be no change in the motion of translation, and if the algebraic sum of the moments is zero, there can be no change in the motion of rotation.

It is evident from what has been said that a body acted on by a system of forces in equilibrium must either be at rest or in uniform motion in a straight line.

A body acted on by a system of forces in equilibrium is sometimes said to be in equilibrium.

72. **Stable, Unstable, and Neutral Equilibrium.**—A body at rest* under the action of several forces which balance each other is said to be in equilibrium. If a body in equilibrium receives a *small* displacement, and the forces acting on it tend to restore it to its original position, it is said to be in **stable equilibrium.**

If, however, when the body receives a small displacement, the forces acting on the body tend to increase the displacement, and

* Or in uniform motion in a straight line.

so to displace the body further from its original position, the body is said to be in **unstable equilibrium.**

In some cases when the body is displaced the forces acting on it tend neither to restore the body to its original position nor to displace it further from this position; that is, the equilibrium of the body is not disturbed by the displacement. In any such case the body is said to be in **neutral equilibrium.**

Thus, a cone resting on a plane horizontal surface, under the action of its weight and the resistance of the plane, is in stable equilibrium if it rests on its base, in unstable equilibrium if it is balanced on its apex, and in neutral equilibrium if it rests on its side.

The equilibrium of heavy bodies is more fully considered in Arts. 79 and 80.

73. **Equilibrium of Two Forces acting at a Point.**— If two forces act at a point the obvious condition of equilibrium is that the forces must be equal in magnitude, and must act in opposite directions.

It follows from this that two forces, acting at different points, will be in equilibrium if they are equal in magnitude and act in opposite directions *along the same straight line.*

It will be seen, too, that if a number of forces act along the same straight line, and the magnitudes of forces acting in opposite directions be distinguished by difference of sign, the forces will be in equilibrium if the algebraic sum of their magnitudes is zero.

74. **Equilibrium of Three Forces acting at a Point.**— Three forces acting at a point will be in equilibrium if any one of them is equal in magnitude and opposite in direction to the resultant of the other two.

It follows from this that if three forces acting at a point are in equilibrium their lines of action must all be in the same plane. The resultant of any two of the forces acts at the same point as the forces, and its line of action is in the same plane as the lines of action of the two forces of which it is the resultant. Hence, if the third force is opposite in direction to this resultant its line of action must also lie in the same plane as the two other forces.

It will be seen, too, that if three forces acting at three different points are in equilibrium the lines of action of the forces must all meet at the same point, and must all lie in the same plane. For, if the three forces are in equilibrium any one of them and the resultant of the other two must be equal in magnitude, and must act in the opposite directions along the same straight line. But the line of action of the resultant of any two of the forces lies in the same plane as the lines of action of the forces, and passes through the point at which these lines intersect. The line of action of the third force must, therefore, also lie in the same plane as the lines of action of the two other forces, and must pass through the point at which these lines intersect. That is, the lines of action of the three forces meet at one point, and lie in one plane.

Hence, if any three forces are in equilibrium they may be considered as acting at a point, for their lines of action meet at a point at which the forces may be supposed to act.

If three forces acting at a point are represented in magnitude and direction by lines drawn in order from any starting point, the three lines so drawn must form a closed figure. For, if the forces are in equilibrium the resultant is of zero magnitude, and the closing line, which represents the resultant in the representation diagram, must therefore be of zero length. That is, the end-point of the diagram must coincide with the starting point, and the lines of the diagram must, therefore, form a *closed figure*.

In this case the closed figure will be a triangle, and the result here considered leads to the theorem, known as **the triangle of forces**, and its converse.

The triangle of forces theorem states that if three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order they must be in equilibrium.

The converse of this theorem states that if three forces acting at a point are in equilibrium they can be represented in magnitude and direction by the sides of a triangle taken in order.

Since all triangles, whose sides are parallel to three given directions, must be *similar triangles,* it follows from the converse theorem that if three forces acting at a point are in equilibrium, and a triangle can be found whose sides represent the forces in direction, they must also represent them in magnitude.

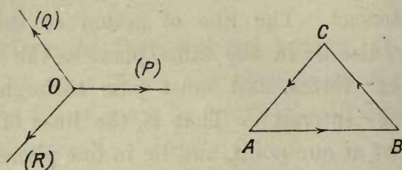

Fig 84.

That is, if a triangle can be found whose sides are parallel to the lines of action of the forces, the lengths of the sides must be in the same ratio as the magnitudes of the forces.

Thus, if three forces, P, Q, and R, acting at a point O (Fig. 84), are in equilibrium, and a triangle ABC is drawn with its sides AB, BC, and CA parallel respectively to the lines of action of the forces P, Q, and R, then the lengths of the sides AB, BC, and CA must be in the same ratio as the magnitudes of the forces P, Q, and R. That is, we must have

$$AB : BC : CA :: P : Q : R.$$

This result is of great service in solving simple statical problems relating to the equilibrium of three forces whose lines of action are not parallel.

Examples.—1. A heavy particle C is suspended by two threads, AC and BC, from points A and B, 5 feet apart on a horizontal line. If the threads AC and BC are respectively 3 feet and 4 feet in length, and the particle weighs 10 grammes, find the tension in each thread.

Let A and B (in Fig. 85) represent the two points from which the particle at C is suspended by the threads AC and BC. The three forces acting on the particle at the point C are its weight, W, acting vertically downwards, the tension T_1 in the thread AC, and the tension T_2 in thread BC. Through the point C draw CD vertically upwards, and through A draw AD parallel to CB. It will now be seen that the sides DC, CA, and AD of the triangle DCA are parallel respectively to the lines of action of the forces W, T_1, and T_2.

It follows, therefore, that the magnitudes of these forces are proportional to the lengths of the corresponding sides.

Since the sides AB, BC, and CA of the triangle ABC are respectively 5 feet, 4 feet, and 3 feet in length, it follows that the angle BCA is a right angle.

It can now be easily proved that the triangle ADC is similar to the triangle ABC, and that its sides DC, CA, and AD are in the ratio 5 : 4 : 3.

Since the three forces acting at C are in equilibrium, we get at once by the triangle of forces that

$$W : T_1 : T_2 :: 5 : 4 : 3.$$

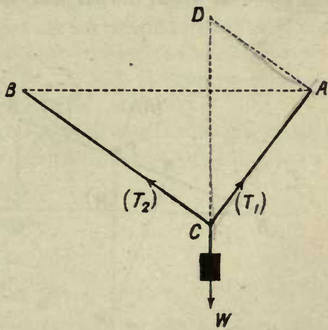

Fig. 85.

The value of W is given as 10 gramme-weights, so that we have

$$10 : T_1 :: 5 : 4$$

or $$T_1 = 8$$

and $$10 : T_2 :: 5 : 3$$

or $$T_2 = 6$$

That is, the tension in the thread AC is equal to the weight of 8 grammes, and the tension in BC to the weight of 6 grammes.

This problem might have been worked by noticing that the sides AB, BC, and CA of the triangle ABC, are respectively *perpendicular* to the lines of action of the forces W, T_1, and T_2. Hence, if we suppose the triangle rotated through 90° in its own plane its sides would be parallel to the lines of action of the forces, and the lengths of the sides would be proportional to the magnitudes of the forces. That is, we have

$$AB : BC : CA :: W : T_1 : T_2,$$

or

$$W : T_1 : T_2 :: 5 : 4 : 3,$$

as obtained above.

The construction given above is, however, more general, and can be applied to any similar problem.

2. A heavy particle rests on a plane inclined at an angle of 30° to the horizontal, and is kept in position by a thread parallel to the plane. If the reaction of the plane be supposed to be at right angles to the plane, find its magnitude; find, also, the tension in the thread.

Let AB (in Fig. 86) represent a vertical section of the inclined plane on which the particle rests at the point P.

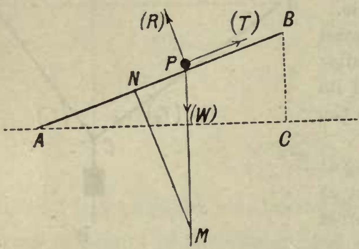

Fig 86.

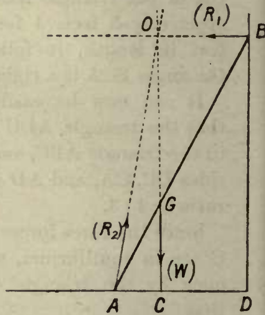

Fig. 87.

The three forces acting on the particle at P, as shown in the figure, are its weight, W, acting vertically downwards, the reaction R acting at right angles to the plane, and the tension T in the thread acting upwards in a vertical plane (the plane of the figure), and parallel to the inclined plane.

Through P draw a line vertically downwards and through any point, M, on this line; draw MN at right angles to AB.

It will be seen that the sides PM, MN, and NP of the triangle PMN are parallel respectively to the lines of action of the forces W, R, and T, and that the lengths of these lines are, therefore, proportional to the magnitudes of the forces they represent.

Since the line AB makes an angle of 30° with the *horizontal*, the line MN must make an angle of 30° with the vertical. That is, the angle PMN in the triangle PMN is 30°, and the sides PM, MN, and NP of the triangle must, therefore, be in the ratio $2 : \sqrt{3} : 1$.

We, therefore, have

$$W : T :: 2 : 1,$$

or

$$T = \frac{W}{2},$$

and

$$W : R :: 2 : \sqrt{3},$$

or

$$R = \frac{W\sqrt{3}}{2}.$$

That is, if W denote the weight of the particle, the tension of the thread is $\frac{W}{2}$, and the reaction of the plane is $\frac{W\sqrt{3}}{2}$, or ·866 W.

[Construct the triangle ABC by drawing a horizontal line through A and a vertical line through B to cut the horizontal through A at the point C. It will be seen at once that this triangle is *similar* to the triangle MPN, and that the forces W, T, and R are proportional, therefore, to the lengths of the sides AB, BC, and AC.

The side AB is usually called the *length*, the side BC the *height*, and the side AC the *base* of the *inclined plane* AB. We, therefore, have

$$\frac{T}{W} = \frac{Height}{Length},$$

and

$$\frac{R}{W} = \frac{Base}{Length}.$$

Since these ratios are readily determined from the triangle ABC when the inclination of the plane to the horizontal is known, the triangle MPN need not be used in solving problems of this type.]

3. A ladder, weighing 24 pounds, rests on a horizontal floor against a vertical wall. The foot of the ladder is 6 feet from the wall, and the top rests on the wall at a height of 12 feet above the floor. If the centre of gravity of the ladder is at a distance of one-third the length of the ladder from its lower end, and if the reaction of the wall is assumed to act at right angles to the face of the wall, find the reactions at the wall and at the floor.

In Fig. 87 let AB represent the ladder resting on the floor at A and against the wall at B, and let G denote the position of its centre of gravity. The three forces acting on the ladder are its weight, W, acting vertically downwards at G, the reaction of the wall, R_1, acting at B at right angles to the face of the wall, and the reaction of the floor, R_2, acting at A.

The three forces are in equilibrium, so that their lines of action must meet at one point. The directions of the forces W and R_1 are known, and the point O at which their lines of action intersect can be determined. The direction of the force R_2 is thus found to be along AO, for its line of action must also pass through O.

The three forces W, R_1, and R_2 may thus be supposed to act at the point O. From O produce the line of action of the force W to cut the horizontal line through A at the point C.

Then, since the lines OC, CA, and AO of the triangle AOC are parallel respectively to the lines of action of the forces W, R_1, and R_2, the length of these lines must be proportional to the magnitude of the forces.

From the data of the question the length of OC is 12 feet, and the length of AC must be 2 feet, for, since $AG = \frac{1}{3} AB$, we must have $AC = \frac{1}{3} AD$, and AD is known to be 6 feet in length. The side AO is, therefore, $\sqrt{148}$ feet, or 12·165 feet in length. We, therefore, have

$$W : R_1 :: 12 : 2,$$

or

$$R_1 = \frac{W}{6},$$

and

$$W : R_2 :: 12 : \sqrt{148}$$

or

$$R_2 = \frac{W\sqrt{148}}{12}.$$

Hence, since the ladder weighs 24 pounds, we have W = 24,

$$R_1 = \frac{24}{6} = 4,$$

and

$$R_2 = 2\sqrt{148} = 24{\cdot}33.$$

That is, the reaction at the wall, R_1, is equal to the weight of 4 pounds, and the reaction at the floor, R_2, is equal to the weight of $2\sqrt{148}$, or 24·33 pounds. The direction of R_2 is seen from the figure to be such that it makes an angle equal to the angle AOC with the vertical on the same side of the vertical as the ladder. That is, the direction of R_2 makes with the vertical an angle whose tangent is $\frac{1}{6}$.

75. **Equilibrium of a Number of Forces acting at a Point.**—If a number of forces acting at a point are in equilibrium their resultant must be of zero magnitude. Hence, if the forces be represented in magnitude and direction by lines

drawn in order from any starting point, the lines so drawn must, as explained above, form a closed figure. In the case of forces whose lines of action all lie in one plane (Art. 61), this closed figure will be a polygon, so that the condition for the equilibrium of a number of co-planar forces *acting at a point* is usually stated in a theorem known as the **polygon of forces.**

This theorem states that if a number of co-planar forces acting at a point can be represented in magnitude and direction by the sides of a closed polygon taken in order the forces must be in equilibrium.

The converse of the theorem states that if a number of co-planar forces acting at a point are in equilibrium, they can be represented in magnitude and direction by the sides of a polygon taken in order.

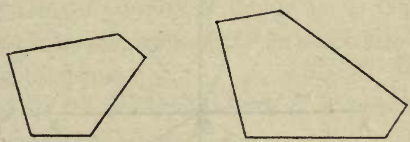

Fig. 88.

It is important to notice, however, that it cannot be said, as in the case of the triangle of forces, that if the sides of a polygon are parallel respectively to the lines of action of the forces, the lengths of the sides are proportional to the magnitudes of the forces. This evidently cannot be true, for polygons whose sides are parallel are not necessarily similar figures. For example, the two polygons shown in Fig. 88 have their sides parallel, but they are evidently not similar figures.

The converse of the polygon of forces is thus much less useful than the converse of the triangle of forces in the solution of statical problems.

76. **General Condition for the Equilibrium of Forces acting at a Point.**—If a number of forces acting at a point O,

as in Fig. 89, are resolved into rectangular components along two straight lines, X′OX and YOY′, drawn at right angles to each other through the point O, and the algebraic sum of the components along each of the straight lines is zero, the forces will be in equilibrium.

For, if X denote the algebraic sum of the components along X′OX and Y denote the algebraic sum of the components along Y′OY, then the resultant of the system of forces is given by

$$R^2 = X^2 + Y^2,$$

and, in order that R may be zero, it is evidently necessary that X and Y should each be of zero value. The quantities X^2 and Y^2 being squares are necessarily of positive sign, so that their sum can be zero only when each quantity is of zero value.

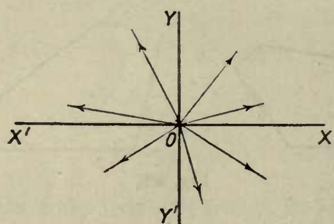

Fig. 89.

77. Equilibrium of a System of Co-planar Forces acting at Different Points in the Plane.—A system of co-planar forces acting at different points in the plane will be in equilibrium if the algebraic sum of the moments of the forces is zero about *every point* in the plane.

It has been explained in Art. 65 that a system of co-planar forces must either be in equilibrium or must reduce to a single resultant force or a couple.

If the system reduces to a single resultant force, the algebraic sum of the moments will have different values for different points in the plane, and will be of zero value only for points on the line of action of the resultant.

If the system reduces to a couple, the algebraic sum of the forces will have the same value for all points in the plane and will not be of zero value for any point.

It follows, therefore, that, if the algebraic sum of the moments of the forces is zero about every point in the plane, the system must be in equilibrium.

In order to prove, in any given case, that a system of co-planar forces is in equilibrium, it is evidently sufficient to show that the algebraic sum of the moments is zero about any three points not in a straight line. For, if the system has a resultant, these three points must lie on its line of action. This, however, is impossible, for the three points are not in a straight line. The system cannot, therefore, have a resultant.

It should be noted, as the converse of what is stated above, that, if a system of co-planar forces is in equilibrium, the algebraic sum of the moments of the forces is zero about any and every point in the plane.

The conditions for the equilibrium of a system of co-planar forces may be stated in another way. Let each force of the system be resolved into two components at right angles along two given directions in the plane, then the system will be in equilibrium if—

(1) The algebraic sum of the components along each of these two directions is zero, and

(2) The algebraic sum of the moments of the forces about any one point in the plane is zero.

If the first of these two conditions is fulfilled, the system cannot, as explained in Art. 76, reduce to a single resultant; and if the second condition is fulfilled, the system evidently cannot reduce to a couple. The system must, therefore, be in equilibrium.

78. **Equilibrium of Parallel Forces.**—A co-planar system of parallel forces is merely a special case of the general co-planar system considered in Art. 77. The system will evidently be in

equilibrium if the algebraic sum of the forces is zero, and if the algebraic sum of the moments of the forces about any one point in the plane is zero.

The special case of three parallel forces in equilibrium should, however, be noticed.

Let the forces P, Q, and R, shown in Fig. 90, be in equilibrium, and let any transverse line ABC cut their lines of action at the points A, B, and C respectively. Since the three forces are in equilibrium, any one of them may be considered as equal and opposite to the resultant of the other two. Hence, if the force R, reversed and acting at C, be taken as the resultant of the forces P and Q acting at A and B respectively, we have, by Art. 63,

$$\frac{P}{Q} = \frac{CB}{CA}.$$

Similarly, if the force P, reversed and acting at A, be taken

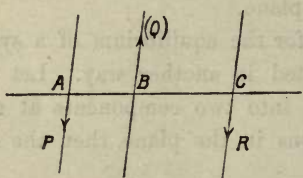

Fig. 90.

as the resultant of the forces Q and R acting at B and C respectively, we have

$$\frac{Q}{R} = \frac{CA}{BA}.$$

It follows at once from these two results that

$$P : Q : R :: BC : AC : AB.$$

That is, when three parallel forces are in equilibrium the magnitude of each force is proportional to the segment intercepted between the lines of action of the other two forces on any transverse line drawn across the lines of action of the forces.

79. **Equilibrium of a Heavy Body Supported at a Point.**—If a body is suspended by a thread attached at a point, or if it is balanced on a pivot point, or if it rests on a plane which it touches at only one point, it is in each case supported by a single force acting at the point of support.

Hence, when a body is supported in this way it is acted on only by its weight and the force supporting it, and if these two forces are in equilibrium they must be equal in magnitude, and must act in opposite directions along the same straight line. The weight of the body acts vertically downwards through the

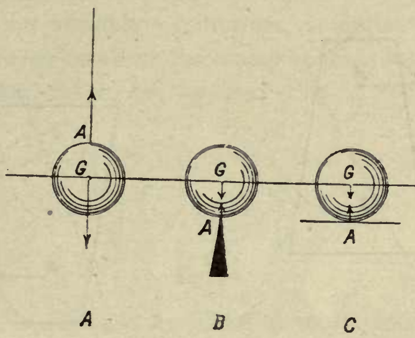

Fig 91.

centre of gravity of the body. The supporting force acting at the point of support, is, therefore, equal to the weight of the body, and must act vertically upwards through the centre of gravity of the body. That is, when the body is in equilibrium the centre of gravity of the body lies on the vertical line through the point of support. When the body is suspended by a thread the centre of gravity is vertically below the point of support, but when the body is balanced on a point, or when it rests on a plane which it touches at only one point, the centre of gravity is vertically above the point of support.

These three cases are shown at A, B, and C (in Fig. 91), where G represents the centre of gravity of the body, and A the point of support.

If we consider a body supported at a fixed point, round which the body is free to move in any direction (thus excluding the case of a body resting on a plane as at C in Fig. 91), it will be seen that the body is in stable equilibrium when its centre of gravity is vertically below the point of support, in unstable equilibrium when the centre of gravity is vertically above the point of support, and in neutral equilibrium when the centre of gravity is the point of support.

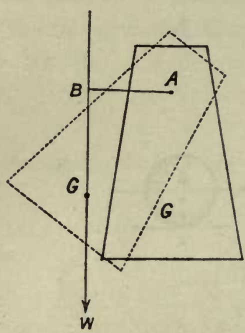

Fig. 92.

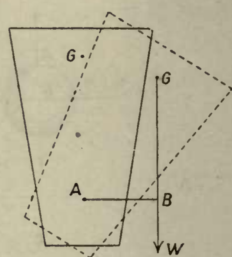

Fig. 93.

Thus, in Fig. 92 it will be seen that if a body in equilibrium with its centre of gravity G vertically below the point of support, A, receives a small displacement into the position indicated by the dotted outline in the figure, the moment of the weight of the body round A, the point of support, tends to restore the body to its original position of equilibrium. That is, the body is in stable equilibrium.

Similarly, it can be seen from Fig. 93 that if a body in equilibrium with its centre of gravity G vertically above the point of support, A, receives a small displacement into the position indicated by the dotted outline in the figure, the moment of the weight of the body acting round A tends to displace the body further from its original position of equilibrium. This body is, therefore, in unstable equilibrium.

When a body is supported at its centre of gravity it is obviously in neutral equilibrium, for the weight of the body must act through the point of support in all positions of the body. That is, the body is in equilibrium in any position.

When a body rests on a plane which it touches at only one point, the point of support changes if the body is displaced, and the stability of its equilibrium depends upon the form of its base. When the base is spherical it can be shown that the body is in stable or unstable equilibrium, according as its centre of gravity is below or above the centre of the spherical base.

Thus, in Fig. 94 it can be seen that if the body shown

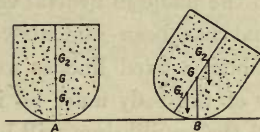

Fig. 94.

in the figure is displaced by tilting it on its spherical base, so that the point at which it rests on the plane changes from A to B, the moment of the weight of the body round B, the new point of support, tends to restore the body to its original position if the centre of gravity is at a point G_1, below C, the centre of the base,* but tends to displace it further from its original position if the centre of gravity is at a point G_2 above C. That is, if the centre of gravity of the body is at a point below the centre of its base the body rests on the plane in stable equilibrium, but if the centre of gravity is above the centre of the base, the body will not in practice stand on this base, but may, in theory, be balanced on it in unstable

* It should be noticed that C will always be on the normal to the plane at the point of support.

equilibrium. An egg, for example, will not stand in stable equilibrium on either end, for its centre of gravity is above the centre of the (approximately) spherical base in each case. It may, however, conceiveably be balanced in unstable equilibrium on either end, but the slightest disturbance would bring it down to its usual position of equilibrium. It should be noticed that this position is one of neutral equilibrium for displacements at right angles to its length, but in stable equilibrium for displacements parallel to its length. In the one case the centre of gravity coincides with the centre of the section of the base in the plane of the displacement, and in the other it is below this point.

It will be seen at once that if the centre of gravity of the body is at the centre of the base, as in the case of a sphere, then the body will rest on the plane in neutral equilibrium.

It follows from what has been said above that when a heavy body, free to rotate about a fixed line as axis, is in equilibrium, the centre of gravity of the body must be in the vertical plane through the axis, and the equilibrium will be stable, unstable, or neutral, according as the centre of gravity of the body is below the axis, above the axis, or on the axis.

80. **Equilibrium of a Heavy Body Standing on a Base on a Plane.**—When a body rests on a plane the area enclosed by a fine thread stretched tightly round the body at the surface of the plane, is called the **base** on which the body rests.

A body resting on a plane on any base will stand in stable equilibrium if the vertical line through its centre of gravity falls within the base. If, however, the vertical through the centre of gravity falls without the base the body will overturn.

Thus, in the case of a cube of any material resting on an inclined plane, the cube will stand in stable equilibrium if the vertical line through its centre of gravity, G, falls within the base ABCD, as in Fig. 95, but will evidently overturn by rotating round the edge AD if this vertical line falls without the base as in Fig 96.

It will be seen from what has been said that a body resting on a horizontal plane may be tilted up, or the plane on which it rests may be tilted up without overturning the body so long as

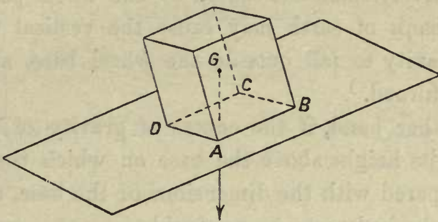

Fig. 95.

the vertical line through the centre of gravity falls within the base on which the body rests. The *stability* of a body resting on any base may thus be considered to depend on the amount of tilting necessary to bring the vertical through the centre of

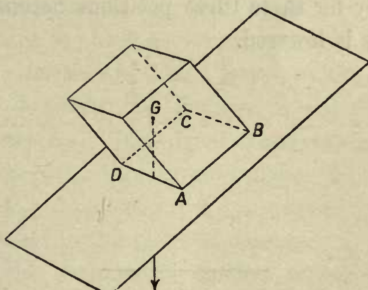

Fig. 96.

gravity of the body to the edge of the base as a limiting position.

If the height of the centre of gravity of a body above the base on which the body rests is large compared with the dimensions of the base, the limits of stability for the body will be comparatively narrow, for under these conditions a comparatively small tilt of the body will cause the vertical line through the centre of gravity to fall outside the base. Thus, if a cart is loaded with a very high load so that the centre of gravity of the cart and load is some distance above the ground, the comparatively small tilt caused by one wheel passing over a stone or a bank of earth may cause the vertical through the centre of gravity to fall outside the wheel base, and the cart may be overturned.

On the other hand, if the centre of gravity of body is low —that is, if its height above the base on which the body rests is small compared with the dimensions of the base, the limits of stability for the body may be very wide.

When a body can rest on a plane on different bases it will be found, by comparing its stability in the different positions, that the lower its centre of gravity is, the wider are the limits of its stability. A brick, for example, can rest on a plane in stable equilibrium on three different bases, and it is easily seen that the limits of stability for these three positions become wider as the centre of gravity is lowered.

CHAPTER XIV.

FRICTION.

81. **The Force of Friction.**—When two plane surfaces are pressed together, and force is applied tending to make one surface move over the other, an opposing force is, in general, set up in the plane of contact of the surfaces in a direction tending to prevent the motion. This force is known as **the force of friction** between the surfaces in contact. It is due to the *roughness* of these surfaces; the small inequalities on one surface engage with the corresponding small inequalities on the other surface, and in this way each surface is able to exert a force on the other in a direction tending to prevent any displacement of one surface relative to the other in the plane of contact.

The force of a friction is thus a stress, as the term is used in Art. 41, acting between the surfaces in the plane of contact parallel to the surfaces. That is, the surfaces exert equal and opposite forces on each other in this plane, the force acting on each surface being directed so as to prevent its displacement relative to the other in the plane of contact.

If either of the surfaces in contact is *smooth* there is no friction between the surfaces, and no force is exerted in opposition to the displacement of one surface over the other. That is, a *smooth* plane surface is unable to exert force on any surface in contact with it in any direction parallel to itself in the plane of contact. It can, therefore, exert force on any body in contact with it only in a direction at right angles to itself.

That is, a smooth surface cannot at any point exert force in a direction *tangential* to the surface at that point, but only in a direction *normal* to the surface at that point.

No real material surface can be so *perfectly smooth* that it offers no resistance tangentially to the motion of another surface over it, but a surface may in practice be so smooth that the resistance, is negligibly small.

The use of oil as a lubricant between metal surfaces in contact tends to reduce greatly the friction between these surfaces.

82. **The Limiting Value of the Force of Friction between Two Plane Surfaces.**—Let a rectangular block of any material be placed as at A in Fig. 97, with one of its plane faces resting on the plane horizontal surface of a plate, B, of the same material. If a force P is now applied to A in a

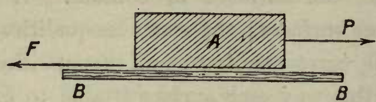

Fig. 97.

horizontal direction, the block A will tend to move by sliding over the plate A. The friction between the surfaces in contact will, however, oppose this tendency to motion, and a force F acting in the plane of contact in a direction opposite to that of P will be established.

If the force P is supposed to be at first very small, the force F will also be very small and equal to P. Then, as P is increased, the value of F also increases in such a way that the two forces are always exactly equal in magnitude and opposite in direction. The force P may be supposed to increase indefinitely, but the force F evidently cannot increase beyond a certain maximum limit determined by the nature of the surfaces in contact and the pressure exerted between them. This maximum limit to the value of the force of friction between the two surfaces, is called the **limiting value** of the force of friction between the two given surfaces under the existing conditions.

When the force P is increased to this limiting value the block A will be on the point of slipping over the plate B, and when P is increased beyond this limit the block will move over the plate under the action of a force (P – F) where F denotes the maximum limiting value of the force of friction between the two surfaces in contact.

Experiment 8.—Get a rectangular block of wood about $6'' \times 3'' \times 2''$ in size, and a plank of the same wood about 3 feet long, 1 foot wide, and 1 inch thick. The surfaces of the block and plank should be plane and even but not too polished or smooth.

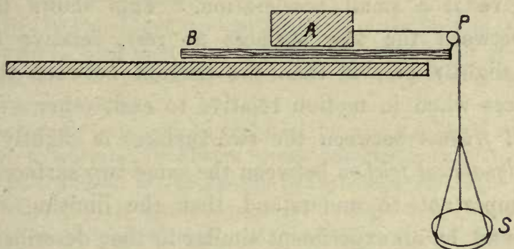

Fig. 98.

Arrange these two pieces of wood on a table, as shown in Fig. 98, so that by means of the pulley at P and the cord carrying the scale pan at S, a force may be applied to the block A in a horizontal direction. This force is applied and measured by the weights placed in the scale pan and evidently tends, as the apparatus is arranged, to make the block A slide over the surface of the plank B.

Begin the experiment by placing a small weight in the scale pan and then go on adding to this weight until the force of friction is no longer able to prevent slipping, and the block A begins to slide over the board on which it rests.

The weight in the pan (together with the weight of the pan itself) when the block A is on the point of slipping is evidently the maximum or limiting value of the force of friction between the two surfaces. For when the block is on the point of slipping the force applied to A must be equal to the maximum force of friction acting in the opposite direction. The force of friction must evidently be at its maximum value, or slipping would not be about to take place.

The point at which slipping is about to take place is a little difficult to determine exactly. The board B should be gently tapped and the weight in the scale pan should be very gradually increased as the slipping point is approached.

It will be found by repeating the experiment a number of times that the value obtained for the limiting fricton is fairly constant.

It will generally be noticed in performing the foregoing experiment that the force, which is only just sufficient to bring the block A to the point of slipping over B, is sufficient, after slipping once takes place, to keep the block in motion and to give it a small acceleration. This shows that the friction between the two surfaces at rest, relative to each other, is slightly greater than the friction between the same two surfaces when in motion relative to each other. That is, the *statical friction* between the two surfaces is slightly greater than the *dynamical friction* between the same two surfaces.

It is important to understand that the limiting value of friction found by an experiment similar to that described above is merely the limiting value in the particular case tested by the experiment. It is obviously to be expected that any change in the conditions under which the friction takes place will change the magnitude of the limiting value.

83. **The Laws of Friction.**—The limiting value of the friction between any two plane surfaces must obviously depend upon the material on which the surfaces are formed and on the degree of roughness of the surfaces. Although these conditions may be specified more or less definitely, they cannot be measured, and cannot, therefore, be involved in any quantitative law. The only measurable quantities on which the friction between the two surfaces may depend are the area of contact of the surface and the pressure exerted normally between the surfaces.

The laws of friction deal, therefore, only with the relations between the limiting value of the friction and these two measurable quantities for any two surfaces.

These relations can be determined only by experiment. The relation between the limiting value of the friction for any two given surfaces, and the area of contact of the surfaces may be determined by the method of the following experiment.

Experiment 9.—Set up the apparatus of Experiment 8 and find the limiting value of the friction between the block and the plank when the block rests on the plank successively on each of the three sides of different area.

In this way the area of contact between the surfaces is varied *without altering the normal pressure between the surfaces*, for the normal pressure is in each case equal to the weight of the block, and is, therefore, constant.

It will be found that the limiting value of the friction is practically the same in each case, and is, therefore, independent of the area of the surface in contact, provided the normal pressure between the surfaces is constant.

The same result will be obtained by using a block and plate of any given material, or a block of one material and a plate of another. That is, the result is true for *any* two specified surfaces.

A block of the dimensions given in Exp. 8 is rather too small for use in this experiment. A block 8″ × 6″ × 4″ will give more consistent results. The surfaces of the block and the plank should be very even and uniform.

The result of this experiment shows, therefore, that the limiting value of the friction between any two surfaces is independent of the area of the surface of contact

The relation between the limiting value of the friction and the normal pressure between the surfaces may be determined by the following experiment.

Experiment 10.—Set up the apparatus of Exp. 8 with the block resting on the face of greatest area.

The normal pressure exerted between the surfaces is, in this case, equal to the weight of the block, but if weights are placed on the block this pressure can evidently be adjusted to any required value *without altering the area of the surface of contact*, and without altering in any way (unless the pressure is made excessive) the nature of the surfaces in contact.

Hence, if we determine the limiting value of the friction, first using the unloaded block, and then the block carrying a number of different loads, we can obtain data from which we can determine the relations between the limiting value and the normal pressure.

It will be found, for any two specified surfaces that the ratio of the limiting value of the friction to the normal pressure between the surfaces is constant.

Example.—In an experiment of this kind the following data and results were obtained :—

Weight of Block.	Load on Block.	Total Weight of Block and Load R.	Limiting Value of Friction F.	Value of Ratio $\frac{F}{R}$.
Grammes-weight.	Grammes-weight.	Grammes-weight.	Grammes-weight.	
1,000	0	1,000	210	·210
	500	1,500	325	·217
	1,000	2,000	425	·212
	1,500	2,500	549	·216
	2,000	3,000	645	·215
	3,000	4,000	850	·212
	4,000	5,000	1,080	·216
Average value of $\frac{F}{R}$ = ·214.				

It will thus be seen that for any two given surfaces the limiting value of friction is directly proportional to the normal pressure between the surfaces. That is, the ratio of the limiting value of friction to the normal pressure between the surfaces is constant.

This is the important quantitative law of friction.

The laws of friction may, therefore, be stated in the following terms.*

* It would be more consistent with actual facts to state these laws as follows :—

The limiting value of the force of friction between any two surfaces is

(1) Directly proportional to the normal pressure per unit area between the surfaces.

(2) Directly proportional to the area of contact of the surfaces.

1. The limiting value of the force of friction between any two surfaces is directly proportional to the normal pressure exerted between the surfaces.

2. The limiting value of the force of friction between any two surfaces is independent of the area of contact of the surfaces.

The dynamical friction between two surfaces in relative motion may evidently depend upon the velocity of the motion. If in an experiment similar to those described above the weight in the pan is made great enough to set the block in motion with acceleration, it is found that the acceleration is uniform. This shows that for the limited range of velocity possible in an experiment of this kind, the limiting value of the friction is constant, and practically independent of the velocity.

84. **The Coefficient of Friction.**—It has been explained in the foregoing article that the limiting value of the friction between any two surfaces is directly proportional to the normal pressure between the surfaces, and that the ratio of the limiting value of the friction to the normal pressure is, therefore, constant for two given surfaces. This ratio is called the coefficient of friction for the two given surfaces.

That is, **the coefficient of friction** for any two specified surfaces is the ratio of the limiting value of the friction between these two surfaces to the normal pressure between the surfaces. Hence, if F denote the limiting value of the force of friction, and R the normal pressure between the surfaces, we have

$$\frac{F}{R} = \text{Coefficient of friction},$$

or, as the coefficient of friction for any two surfaces is generally denoted by μ, we have

$$\frac{F}{R} = \mu,$$

or

$$F = \mu R.$$

The experimental determination of the coefficient of friction for any two given surfaces evidently involves the determination of F, the limiting value of the friction corresponding to any convenient value of R, the normal pressure between the surfaces.

The determination can, therefore, be made conveniently by the method of Experiment 10. This experiment is, in fact, a determination of the coefficient of friction for the surfaces used, and the example given at the end of the experiment shows how the value of the coefficient of friction can be calculated from the data of the experiment. The values of the ratio F/R, given in the last column of the table in the example, are values of μ, the coefficient of friction for the surfaces to which the data apply, and the mean value of the ratio given at the bottom of the table is the mean value given by the experiment of the coefficient of friction for these surfaces.

The determination may also be made by the method of the following experiment.

Experiment 11.—Take a block and plate similar to that used in Experiment 10, and place the block on the plate so that the surfaces for which the coefficient of friction is required are in contact. If necessary, one face of the block and the upper surface of the plate may be coated or covered with the surfaces to be tested.

Place the plate and block in a table, and tilt the plate gradually by raising one end until the block is on the point of slipping down the inclined surface of the plate.

The coefficient of friction for the two surfaces is then given by the tangent of the angle at which the surface of the plate is inclined to the horizontal when slipping is about to take place.

For, let PQ in Fig. 99 denote the position of the plate when the block A is on the point of slipping down the plane. The forces acting on the block are its weight, W, acting vertically downwards through its centre of gravity, the limiting friction F acting up the plane, and the normal reaction of the plane R acting outwards at right angles to the plane.

Let PN, the horizontal line through P, and QN, the vertical line through Q, meet at the point N.

The three forces, P, Q, and R, whose lines of action meet at the point O, are in equilibrium by the triangle of forces, as explained in Example 2 in Art. 74, and we therefore have

$$\frac{F}{R} = \frac{QN}{PN},$$

or

$$\mu = \frac{QN}{PN}.$$

Hence, if the distances QN and PN are carefully measured for the position at which the block A is on the point of slipping, the ratio of the two distances, taken as above, gives the coefficient of friction for the two surfaces in contact.

It will be seen that

$$\frac{QN}{PN} = \tan QPN.$$

That is, if α denote the angle at which the plane PQ is inclined to the horizontal when slipping is about to take place, we have

$$\frac{F}{R} = \mu = \tan \alpha.$$

Hence, if instead of measuring the distances QN and PN, the angle

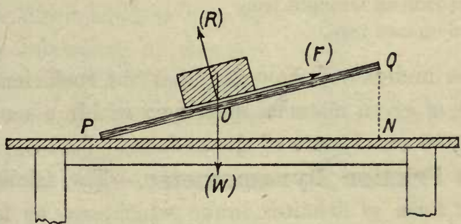

Fig. 99.

NPQ or α is measured, the coefficient of friction, μ, is given by the relation,

$$\mu = \tan \alpha.$$

For this reason, α, the angle whose tangent is μ, the coefficient of friction, is sometimes called the angle of friction.

More accurate results can, however, be obtained in simple experiments by using a fairly long plane, PQ, and measuring the distances QN and PN to determine μ.

Example.—In an experiment similar to that described above three separate determinations of the slipping point were made, and the distances QN and PN were measured for each determination.

The following data were thus obtained :—

QN (mm.).	PN (mm.).	$\mu = \frac{QN}{PN}$.
243	962	·251
246	961	·256
240	963	·249
Mean value of μ,	.	·252

The values of the coefficients of friction in a few common cases are given below.

Surfaces in Contact.	Values of μ.
Hardwood on hardwood; with the grain, fibres parallel,	0·50
Hardwood on hardwood; with the grain, fibres at right angles, ...	0·33
Hardwood on hardwood; across the grain, . . .	0·26
Metal on hardwood, ...	0·55
Wrought iron on wrought iron, ...	0·18
Cast iron on cast iron, ...	0·15

It will be understood, however, that the coefficient of friction for surfaces of given material must vary within a somewhat wide range with the exact state of the surfaces.

85. **The Friction Dynamometer.**—The friction dynamometer is a form of friction brake which can be applied to a pulley or flywheel driven by an engine, in order to measure the horse-power of the engine.

A simple form of this type of dynamometer is shown in Fig. 100. It consists of a belt or rope, AB, passed over the pulley P in the manner shown in the figure. One end of the belt at A is attached to a strong spring-balance S, and the other end at B carries a weight W, which can be adjusted to any desired value.

The pulley is driven by the engine in the direction indicated by the arrow, so that the friction belt is urged by the friction between it and the pulley in the direction AB. The force of friction between the belt and the pulley acts everywhere along the tangent to the circumference of the pulley, and must, therefore, act on the belt at every point in a direction parallel to its length and tending to pull it round from A towards B.

It follows from this that the pull on the spring-balance S will be greater than the weight W by the force of friction exerted by the pulley on the belt. Hence, if P denote the pull exerted on the spring-balance, and F the force of friction between the belt and the pulley, we have

$$P = F + W,$$

or

$$F = P - W.$$

The value of F can thus be determined if the values of P and W are known. It can also be increased or diminished as may be required by increasing or decreasing the value of W. Thus, when W is increased P also increases, and as the pressure of the belt on the pulley is in this way increased the value of F must also increase. Similarly, when W is decreased, the values of P and F also decrease.

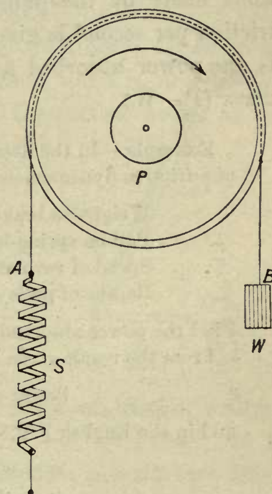

Fig. 100.

In order to use this brake to measure the power of an engine or motor the pulley is driven by the engine, and the weight W on the brake is adjusted until the engine is found to be exerting its full power and working generally under the conditions under which it is to be tested.

The reading of the spring-balance at S, and the rate of revolution of the pulley (given by a speed indicator), are then carefully noted and recorded.

Let P denote the pull indicated by the spring-balance, and W the weight on the brake, then, as above, we have

$$P - W = F.$$

Now, during each complete revolution of the pulley this force, F, the force of friction exerted by the belt on the pulley, is overcome through a distance equal to the circumference of the pulley in the direction in which the force acts. Hence, if r denote the radius of the pulley, and n the number of revolutions made by the pulley per second, the work done against friction per second is given by $F \cdot 2\pi nr$ or $2\pi nr(P-W)$. That is, the **power** absorbed and measured by the brake is given by $2\pi nr(P-W)$.

Example.—In the determination of the power of a motor by means of a friction dynamometer the following data were obtained:—

Weight on brake, . .	10 pound-weights.
Pull on spring-balance, .	16 ,,
Speed of revolution, . .	1,200 revs. per minute.
Radius of pulley, . .	6 inches.

Find the power absorbed by the brake in horse-power.

From the result given above, we have—

$$\text{Power abstracted} = 2\pi nr(P - W),$$

and in the English F.P.S. system we have—

$$n = 20 \text{ (revs. per second).}$$
$$r = \cdot 5 \text{ (foot).}$$
$$P = 16 \text{ (pound-weights).}$$
$$W = 10 \text{ (,,).}$$

The power absorbed is therefore

$$(2\pi \times 20 \times \cdot 5 \times 6) \text{ ft.-pounds per second,}$$

or $$\frac{120\pi}{550} \text{ horse-power,}$$

or $$\cdot 685 \text{ horse-power.}$$

86. **Reaction of a Rough Plane Surface.**—The reaction of a *smooth* plane surface against any body pressing on it can act

only in a direction normal to the surface. For, if we suppose the reaction at any point P, Fig. 101, on the surface to act in the direction PR, making an angle NPR, with the normal PN, we can resolve the reaction into a normal component along PN, and a component parallel to the surface along PM. But, if the surface is smooth its reaction at any point cannot have a component parallel to the surface, for a smooth surface cannot exert force in a direction parallel to itself

The reaction of a smooth surface on any body can act, therefore, only along the normal to the surface.

The reaction of a rough plane surface may, however, act in a direction inclined at an angle to the normal, for it is, in general, the resultant of the normal reaction of the surface acting as in

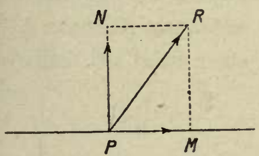

Fig. 101.

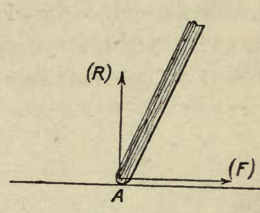

Fig. 102.

the case of a smooth surface, and the force of friction acting parallel to the surface.

Thus, when a heavy rod is set up against a wall, with one end resting on the ground, the reaction of the ground, as shown at A in Fig. 102, is the resultant of the normal reaction R, acting vertically upwards, and the frictional reaction F, acting parallel to the ground in the proper direction to prevent slipping. The magnitude of F will be only just sufficient to prevent slipping, and may, therefore, have any value between zero and its maximum limiting value.

Whatever value F may have within these limits, the direction of the resultant reaction of the surface at A is such that it makes an angle with the normal at A whose tangent is F/R.

The size of this angle depends upon the value of F, and may have any value between zero, when F is zero, and a certain maximum value when F has its maximum limiting value.

If the rod is on the point of slipping F will have its limiting value for the two surfaces in contact, and the ratio F/R will be the coefficient of friction for the surfaces.

The *greatest angle* which the direction of the reaction at A can make with the normal is, therefore, the angle whose tangent is F/R, where F has its maximum limiting value, and $F/R = \mu$, where μ is the coefficient of friction for the surfaces in contact. That is, the greatest angle which the reaction of a rough surface can make with the normal to the surface is the angle of friction.

CHAPTER XV.

THE BALANCE.

87. **Theory of the Balance.**—The general construction of a simple form of balance has already been described in Art. 23. The elementary theory of its construction and action can now be considered.

Let A, B, and C, in Fig. 103, represent sections of the knife-edges of the beam of a balance in a vertical section taken lengthwise through the beam at right angles to the edges. The edges at A and B carry the scale-pans, and the edge at C is that on which the beam rests and turns.

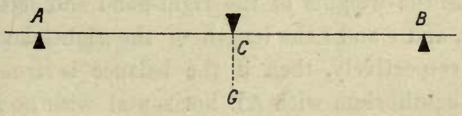

Fig. 103.

These knife-edges are fixed on the beam, so as to be exactly parallel to each other at right angles to the length of the beam, and they are usually set so as to lie truly in the same plane. The central knife-edge at C is fixed exactly midway between the two edges at A and B, so that the line AB is bisected at C; and the two *arms*, CA and CB, are exactly equal.

In order that the beam may set in stable equilibrium with the line AB horizontal when balanced, without the scale-pans, on the knife-edge at C, the centre of gravity of the beam and everything *rigidly* attached to it must be at a point G below C on the line CG, drawn through C at right angles to AB.

In the case of a beam constructed in this way, it is clear that if equal weights are suspended from the knife-edges at A and B their resultant must act at the knife-edge at C, and cannot, therefore, disturb the equilibrium of the beam. This can also be seen by taking moments round the knife-edge at C; for, if the weights at A and B are equal, their moments round C must be equal and opposite, since the arms CA and CB are equal.

Hence, if scale-pans of *equal weight* are suspended from the knife-edges at A and B, the beam will still set in stable equilibrium with the line AB horizontal as before, and it will always set in equilibrium in this position when the weights in the pans are exactly equal.

A balance which fulfils this condition is said to be **true.**

It should be noticed that it is essential in order that a balance may be true (1) that the arms of the beam should be equal, and (2) that the scale-pans should be of equal weight. For, let R and L denote the weights of the right-hand and left-hand pans respectively, and r and l the length of the right-hand and left-hand arms respectively, then if the balance is true the beam must set in equilibrium with AB horizontal with no load in the pans, and also when the same load W is carried by each pan. We must, therefore, have

$$Rr = Ll,$$

and also $$(R + W)r = (L + W)l.$$

Taking the difference of these two equations, we get

$$Wr = Wl, \text{ or } r = l;$$

and it follows from this, since

$$Rr = Ll, \text{ that } R = L.$$

Hence, in order that a balance may be true, so that it sets in equilibrium with the plane of the knife-edges horizontal when the pans carry equal loads, the arms of the beam must be equal, and the pans must be of equal weight.

Another important characteristic essential to a good balance is *sensibility* or sensitiveness. A balance must weigh truly, but it is even more essential that it should be sensitive to a small difference in the weights in the pans, and should indicate any very small difference of this kind by an appreciable deflection of the beam from its position of equilibrium.

Suppose a balance to carry a weight W in one pan, and a weight $(W + x)$ in the other pan, and let it set in equilibrium when so loaded, with the line AB inclined at an angle α to the horizontal, as shown in Fig. 104.

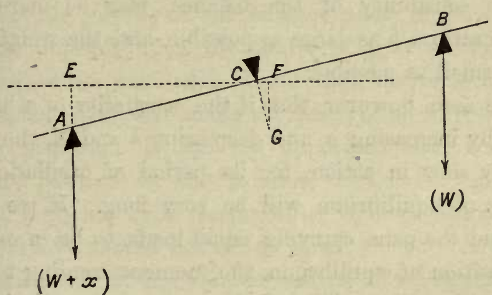

Fig. 104.

Since the beam is in equilibrium in this position the moments of the forces acting on it, taken round the central knife-edge at C, must balance each other. The resultant of the two equal weights, W, acting at A and B must act at C, so that the algebraic sum of their moments round C is zero, and they need not be considered in taking moments round C. It follows, therefore, that for equilibrium we must have the moment of the weight x acting at A round C equal to the moment of the weight of the beam acting at G round C. That is, if X denote the weight of the beam, we must have

$$x \,.\, CE = X \,.\, CF.$$

Now, if a denote the length of the arm CA, and h denote the distance CG, we have

$$CE = a\cos\alpha, \text{ and } CF = h\sin\alpha.$$

We therefore get

$$x \,.\, a\cos\alpha = Xh\sin\alpha,$$

or
$$\tan\alpha = \frac{a}{Xh}\,.\,x.$$

This result shows that, for a given value of x, the value of α depends upon the value of $\frac{a}{Xh}$. That is, for a given small difference, x, between the weights in the pans, the value of α, and the sensibility of the balance, may be increased by making the ratio a/h as large as possible, and the weight of the beam X as small as possible.

It will be seen, however, that if the sensibility of a balance is made high by increasing a and decreasing h and X, the balance will be very slow in action, for its period of oscillation about any position of equilibrium will be very long. If we suppose the beam and the pans, carrying equal loads, to be in oscillation about its position of equilibrium, the moment tending to restore it to its position of equilibrium for any angular displacement, α, from this position is evidently $Xh\sin\alpha$, as explained above. The period of oscillation depends, therefore, as explained in Art. 43, upon the value of $Xh\sin\alpha$, and the moment of inertia of the oscillating system made up of the beam and the loaded pans. If $Xh\sin\alpha$ is small, and the moment of inertia great, the period of oscillation may be very long, so that if X and h are small, and a is comparatively large, the time of swing might be so long that it would be impossible to make a weighing in any reasonable time.

It thus appears that the very conditions which are necessary for high sensibility are those which make the balance impracticably slow in action. It follows, therefore, that in designing and constructing a balance a compromise must be effected between sensibility and quickness of action. This compromise has led to the construction of balances of two different types: *long beam* balances of comparatively slow action, and *short beam* balances of quicker action. A short beam balance is generally provided with a somewhat long pointer moving over a finely divided scale, and in some cases a reading microscope is used for reading the position of the pointer on the scale. In this way very small deflections of the beam from its zero position can be detected, and the working sensibility of the balance is increased.

In connection with what has been said above, it should be noticed that the *stability* of the balance beam in its position of equilibrium depends upon the centre of gravity of the beam, G, being below the knife-edge C, and upon the moment $Xh \sin a$ being sufficiently great to make the beam come to rest always in the same position. It is very important that the zero or equilibrium position of the balance should be constant and invariable, and, for this reason only, it is necessary that the moment $Xh \sin a$ should not be too small.

The reasons for setting the knife-edges of the beam all in one plane can now be considered. It is clear that when the pans carry equal loads the forces acting on the beam at the knife-edges A and B (Fig. 105) are equal, and that the resultant of these two forces acts vertically downwards at a point C midway between A and B. If the central knife-edge is set at this point in the same plane with those at A and B, as in Fig. 103, this resultant can have no moment round it, and it follows that the sensibility of the balance and the moment of the couple tending to restore it to its position of equilibrium ($Xh \sin a$) are quite independent of the load carried by the balance. The time of swing of the balance will not, however, in this case be independent of the load; for, although the moment, $Xh \sin a$, is constant for all loads, the moment of inertia of the swinging system increases with the load, and the time of swing will therefore increase as the load increases.

If, however, the central knife-edge is set, as at C′ in Fig. 105, in the plane bisecting AB at right angles, but above the point C, the beam will set in equilibrium with the points C and G vertically below C′, but the resultant force acting at C will have moment round the knife-edge at C′ when the beam is displaced from its horizontal position of equilibrium. If R denote the magnitude of the resultant acting at C, the moment acting on the beam for a displacement α will evidently be $Xh \sin \alpha + Rh' \sin \alpha$, where h denotes the distance C′G and h' the distance C′C. From this result it will be seen that, under these conditions, the sensibility of the balance will decrease as the load

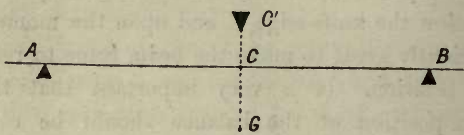

Fig. 105.

increases, and that the change in the time of swing as the load increases will be less marked than when C and C′ are coincident. If the knife-edges were arranged so that C and G coincided, the time of swing would be practically constant.

The case in which the central knife-edge is fixed below the point C need not be considered, for it can be seen that, except under certain evident conditions, a beam with the knife-edge so arranged would be in unstable equilibrium.

It will be seen, therefore, that, in the case of a balance constructed in the usual way, with the knife-edges all in one plane, the sensibility will be practically independent of the load, but the time of swing will increase as the load increases.

One very important essential in the construction of a balance is the *rigidity* of the beam. The beam should be rigid enough to show no appreciable bending under the maximum load it is designed to carry. The girder beams in general use for

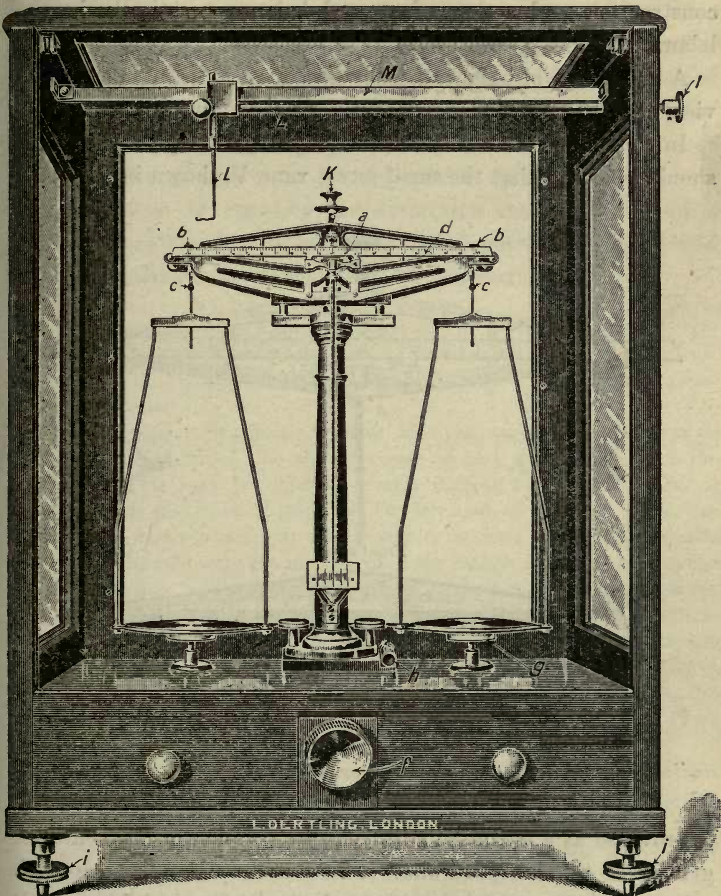

Fig. 106.—*a*, Central knife-edge; *b*, *b*, end knife-edges; *c*, *c*, stirrups for carrying pans; *d*, graduated bar for rider; *f*, milled head for lowering the pan rests and releasing the beam; *g*, pan rests for arresting and supporting the pans; *h*, spirit level; *i*, levelling screws; K, gravity bob; M, *l*, *l*, arrangement for moving the rider.

accurate balances are designed to give the necessary rigidity without making the weight of the beam unduly great.

The general details of the construction of an accurate balance are somewhat complicated, and are best learnt by studying the construction and action of a good balance practically in the laboratory, and by working with it.

A good form of balance is shown in Fig. 106, and an enlarged view of the beam of this balance is given in Fig. 107.

In connection with the theoretical discussion given above, it should be noted that the small screw vane V, shown in Fig. 107,

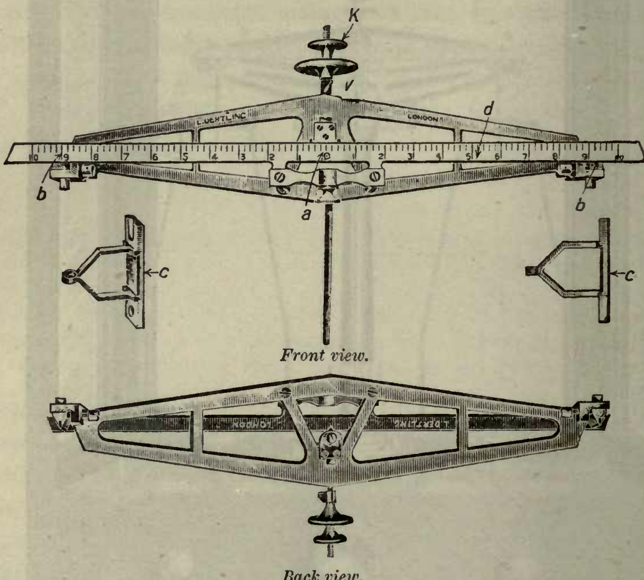

Fig. 107.

is provided as a means of adjusting the position of the centre of gravity of the beam so that it lies, as explained above with reference to Fig. 103, below C on the line CD, which bisects AB at right angles at C. It will be seen that by turning the vane to the right or to the left, the position of the centre of gravity is moved very slightly in the same direction. The gravity bob K, shown in the same figure, is provided for raising or lowering the centre of gravity of the beam by screwing the bob up or down. It thus acts as a fine adjustment for adjusting the sensibility of the balance.

88. **Practical Determination of the Sensibility of a Balance.**—The sensibility of a balance is usually determined by finding the difference in load which will deflect the pointer attached to the beam through one division on the scale over which it moves.

Thus, if it is required to determine the sensibility of a balance when the load in each pan is 20 grammes, the method of the following experiment might be adopted.

Experiment 12.—Find the sensibility of the given balance for a load of 20 grammes in each pan. Set the beam of the balance free, and see if it swings freely and regularly, and is generally in proper adjustment.

Then place a 20 gramme load in each pan, and find the division on the scale at which the pointer comes to rest with this load on the pans. This may be done by simply waiting until the beam stops swinging, and then reading off the division of scale at which the pointer comes to rest. It can, however, be done much more expeditiously by following the movement of the pointer over the scale as the beam swings, and reading the turning points for any three successive swings. If then we take the mean of the first and third readings, and then the mean of this mean and the second reading, we get the reading at which the pointer would come to rest. Thus, if the three observed turning points are at divisions 6, 17, and 8 on the scale,* the pointer would come to rest at the division marked 12 on the scale.

[The reason for this method of taking the mean of the observed readings is readily understood. If the beam in swinging were quite free from friction and air resistance, its swing would obviously be of constant amplitude, and the resting point could at once be found by taking the mean of any two successive turning points. On account, however, of the damping effect of friction and air resistance, the swings gradually decrease in amplitude, and in order to find

---

* The scale is supposed to be numbered from one end, not from the middle, as it sometimes is. The need for plus and minus signs to distinguish between right and left readings is thus avoided.

the resting point by observing the turning points it is necessary to take the mean in such a way as to eliminate the decrement due to damping. If this decrement is small it may be supposed to be the same for several consecutive swings, so that if we denote it by x it will be seen that with no damping the pointer would swing between two constant turning points at, say, the pth and qth divisions on the scale; but with damping, the successive turning points for a small number of swings would be approximately at the divisions p, $(q-x)$, $(p+2x)$, $(q-3x)$, $(p+4x)$, &c., on the scale.

Now, if we take any three consecutive turning points, such as $(q-x)$, $(p+2x)$, and $(q-3x)$, it will be seen that the mean of the first and third is $(q-2x)$, and that the mean of this mean and the second is the mean of $(q-2x)$ and $(p+2x)$, or the mean of p and q which is obviously the true resting point.

It will be seen that the same result is obtained by taking any odd number of successive turning points, and taking the mean, first of those for swings to the right, then of those for swings to the left, and, finally, the mean of the two means so obtained.]

Having found the resting point with a load of 20 grammes in each pan, now place a small weight, say, a milligramme, in one pan, so as to make a difference of one milligramme in the weights carried by the pans, and find again the reading on the scale at which the point comes to rest. Suppose this resting point to be at 7 on the scale.

It follows from these data that a difference of 1 milligramme in the loads on the pans changes the resting point of the pointer through 5 divisions on the scale.

That is, a difference of ·2 milligramme between the weights in the pans gives a deflection of 1 division on the scale. This is the sensibility of the balance when the load on the pans is 20 grammes.

We can find in the same way the sensibility of the balance for different loads, from no load to the full load the balance can carry.

It will be found, as a rule, that the sensibility is practically the same for all loads. It generally decreases slightly as the load is increased.

From the data obtained a curve may be plotted showing how the sensibility varies, in the case of the given balance, with the load.

The accuracy of this experiment depends upon the care with which the different resting points are determined. Each resting point should be determined as the mean of several consistent determinations.

It will be understood that the sensibility of a balance found in the manner explained in this experiment, is a purely empirical quantity, which depends for its absolute value on the length of the scale divisions, and also on the length of the pointer.

When the sensibility of a balance is known the weight of a body may be determined without wasting time in making the final small adjustments of the weights which are usually necessary to bring the beam exactly into its position of equilibrium. If it is found when the adjustment is *nearly* complete that the resting point is n scale divisions from the balancing position, and if s is the sensibility of the balance in milligrammes per scale division, the weight of the body is evidently ns milligrammes greater or less than the weight in the pan, according as the weight pan is lighter or heavier than the other. This method of weighing is sometimes called the method of weighing by vibrations.

89. **The Use of Riders.**—Let the arm of a balance be divided into ten equal parts, and let the divisions be numbered from 0 to 10 outwards from the centre, the division marked 0 being at the central knife-edge, and the division marked 10 at a terminal knife-edge. The distance of any one of these divisions from the central knife-edge is thus equal to a certain number of tenths of the length of the arm, and it follows at once, by the principle of moments, that if a given weight is carried by the beam at that division, it is equivalent only to a certain number of tenths of its real weight placed in the scale-pan. Thus, if a centigramme is placed on the beam at the division marked 3, it is equivalent to 3 milligrammes in the scale-pan, or if placed at the division marked 8, it is equivalent to 8 milligrammes in the scale-pan. Hence, if a piece of platinum wire, weighing exactly one centigramme, is bent into the form shown in Fig. 108, so that it can be placed as a **rider** at any point on the divided beam, the use of milligramme weights can be dispensed with, for any weight smaller than a centigramme can be obtained by adjusting the position of the rider on the beam. If each of

the ten divisions on the arm is further subdivided into ten divisions, the equivalent weight of the rider at any point on the arm can evidently be obtained in milligrammes and tenths of a milligramme, by simply reading the position of the rider on the scale marked along the divided arm. Thus, if the rider is placed at the 63rd division on this scale it is equivalent to 6·3 milligrammes in the scale-pan.

The process of weighing with a rider thus resolves itself into the following procedure. The weighing is first made to the nearest centigramme by placing weights in the pan in the usual way. The position of the rider on the beam is then adjusted

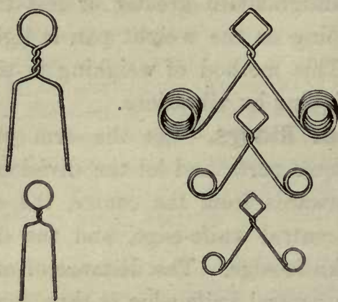

Fig. 108.

until an exact balance is obtained. The required weight is then obtained from the weights in the pan and the position of the rider. Thus, if the weight in the pan is found to be 3·54 grammes, and the rider is at division 36 on the beam scale, the weight required is 3·5436 grammes.

It is sufficient for most purposes if the beam is divided only along one arm on the same side as the pan in which the weights are usually placed in weighing. It is usual, however, to divide the beam along the whole length between the terminal knife-edges, so that the weight equivalent of the rider may be added to, or taken from, the weight in either pan, as may be convenient.

In most balances a special bar is attached to the beam for the purpose of carrying the rider and special appliances are provided for putting on and taking off the rider without opening the balance-case or disturbing the beam. In Fig. 106 the rider bar is shown at RR, and the lever for lifting and carrying the rider is shown at L.

The general details relating to the use of a rider on an accurate balance are best learnt by actual practice in weighing.

90. **Special Methods of Weighing.**—In cases where the accuracy of the balance is in doubt, any weighing may be tested by the following special methods.

Place the body to be weighed in one pan, and counterpoise it exactly with fine shot or pieces of wire in the other pan. Then remove the body and put weights in its place until an exact balance is again obtained. By this method the weights must give truly the weight of the body, whether the balance is accurate or inaccurate, provided it is sufficiently sensitive for the purpose. The weights, it will be seen, are placed in the same pan as the body, and they balance the same counterpoise under exactly the same conditions, so that their weight must be exactly equal to the weight of the body.

Another method of weighing which is useful for detecting and eliminating any error that may be caused by the arms of the balance not being exactly equal.

The body is weighed in the ordinary way, first in one pan and then in the other, and the geometric mean of the two weights so obtained is taken as the true weight. When the two weights are very nearly equal, as they always would be in practice, a sufficiently accurate result is obtained by taking their arithmetic mean instead of the geometric mean.

Let P and P′ denote the two weights obtained by weighing the body first in the left pan and then in the right pan, and let l and r denote the lengths of the left and right arms respectively of the balance. Then, if W denote the true weight of the body, and if we assume the balance to be in exact equilibrium when there is no load in the pans, we must have—

$$Pl = Wr,$$

and $$P'r = Wl.$$

From these relations we at once get

$$PP' = W^2,$$

or $$W = \sqrt{PP'}.$$

That is, the true weight W is the geometric mean of the false weights P and P′. When, however, P and P′ are nearly equal, so that $P' = P + \delta$, where δ is small, we have—

$$W = \sqrt{P(P + \delta)} = \sqrt{P\left(1 + \frac{\delta}{P}\right)} = P\left(1 + \frac{\delta}{2P}\right) = P + \frac{\delta}{2}.$$

That is, when P and P′ are nearly equal, the true weight W is approximately equal to their arithmetic mean.

CHAPTER XVI.

GENERAL PROPERTIES OF MATTER.

91. **The Constitution of Matter.**—A piece of matter of any particular kind is supposed to be made up of minute ultimate particles or **molecules** which are assumed to be the smallest particles of that particular kind of matter which can exist independently. If the piece of matter is supposed to be divided and subdivided into smaller and smaller pieces, the ultimate particles into which it can conceivably be divided, and still continue to be matter of the same particular kind, are its molecules. As explained below, a molecule is generally divisible into component parts called **atoms**, but the molecule is the physical unit in the constitution of matter, and any particular kind of matter, whether it be an element or a compound, is supposed to be built up of its molecules and not of its atoms as constituent units.

The molecules which make up any piece of matter are supposed to be aggregated together without being actually in contact. Force is exerted mutually between the molecules and the group of molecules which constitute any portion of matter are held together by these intermolecular forces. It is supposed also that the molecules of a body are not at rest, but in rapid vibratory motion.

It is thus assumed that the molecules of a body are free and distinct from each other, that a stress of attraction or repulsion * exists between each molecule and every surrounding molecule within its range,† and that every molecule is in rapid vibratory motion.

A piece of matter considered as a system of molecules may thus possess *molecular potential energy* in virtue of the configuration of the system, and *molecular kinetic energy* in virtue of the motion of its molecules.

The size of a molecule is almost inconceivably small. We have no exact knowledge of the actual form or size of a molecule, but approximate estimates can be made in several ways of the probable order of magnitude of the diameter of a molecule considered as a small spherical particle.

Some idea of this magnitude may be obtained by considering that a piece of ordinary gold leaf, which is less than four-millionths of an inch in thickness, probably consists of more than a hundred layers of molecules.

Lord Kelvin illustrates the size of a molecule by stating that if a drop of water were magnified to the size of the earth the molecules would be about the size of an orange or a cricket ball.

Although the molecule is the unit in the physical constitution of a piece of matter, the molecule of any substance is itself a group of component parts called **atoms**. Thus a molecule of water is a group of three atoms made up of one atom of oxygen and two atoms of hydrogen. Similarly, a molecule of chalk is a group of five atoms made up of one atom of calcium, one atom of carbon, and three atoms of oxygen.

When the molecules of any substance are made of atoms of

* It has been suggested that the law expressing the stress between two molecules is such that the stress is one of attraction or repulsion, according as the distance between them is greater than or less than a certain small limit.

† If the stress between two molecules decreases very rapidly with increase in their distance apart, it may become negligibly small beyond a certain small range.

different kinds the substance is said to be a *compound substance*, but when the molecules of a substance are made up of one or more atoms of the same kind, the substance is said to be an *elementary substance.* Thus water and chalk are compound substances, but a substance, such as oxygen or hydrogen, whose molecules are made up of two similar atoms is an elementary substance. The atoms of an elementary substance are said to be atoms of that substance, although, strictly speaking, the substance is made up only of molecules.

A molecule is thus supposed to be a group of atoms held together by interatomic forces in much the same way as the molecules of a piece of matter are held together by intermolecular forces.

Until quite recently an atom was considered to be an ultimate and indivisible particle of matter. There is now, however, abundant experimental evidence to establish the theory that an atom is really a group of component particles held together by the stresses existing between the particles. There are probably only two kinds of particles which enter in this way into the constitution of atoms, and it is probable that the atoms of different substances differ from each other only in the number and grouping of their component particles. It thus appears to be possible for the atoms of one substance to change by a process of disintegration and regrouping into the atoms of another substance. That is, any substance in which the atom is a large and complex group may possibly change into a substance in which the atom forms a smaller and simpler group. This process of change from one substance to another may take a very long time or a very short time, and may pass through many well-marked intermediate stages.

The element **radium** derives the interest which at present attaches to it from the fact that its atoms are supposed to be in process of disintegration, and there is satisfactory evidence to show that the element helium is derived from radium, as a result of this disintegration.

92. **States of Aggregation of Matter.**—The three normal states of aggregation of matter are the solid state, the liquid state, and the gaseous state.

In a piece of matter in the **solid state** the molecules are so aggregated together under the control of intermolecular stresses that their relative positions are fixed, and every molecule is able to offer resistance to displacement in any direction from the position it occupies in the body. That is, a molecule may be displaced slightly from the mean position it occupies without causing any rupture in its relations with the surrounding molecules, but the adjustment of the stresses between it and the surrounding molecules is disturbed by the displacement, and a resultant stress which opposes the displacement, and increases as the displacement increases, is thereby set up. Hence, if a force is made to act on any particle in a solid body, the particle may be slightly displaced in the direction of the force, but the displacement sets up in the material around the particle an opposing stress which resists the displacement, and tends to restore the displaced particle to its initial position.

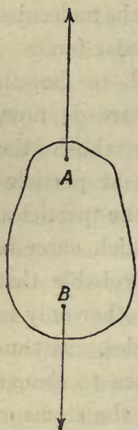

Fig. 109.

If the force applied to the particle is sufficiently great to displace the particle beyond a certain limiting position, the particle *breaks* away from the rest of the material, or becomes the starting point of a fracture in the material.

It will be seen from what has been said, that if two equal forces acting in opposite directions along the same straight line, act on a solid body at points A and B, as in Fig. 109, the material of the body between A and B will be subjected to a tension or a pressure, and, in virtue of its properties as a solid, will be able to sustain and resist this stress unless it exceeds a certain limit. The body will be slightly stretched or compressed by the applied stress, but the internal stress set up in the material by this change in the molecular configuration of the body, will resist and balance the applied stress up to the yielding point of the material.

When a block of any solid material rests on the ground under the action of its own weight, it is evident that any horizontal slice of the block must be subject, in this way, to pressure. The weight of the overlying portion of the block acts vertically downwards on the slice, and the resistance of the underlying portion on which the slice rests acts vertically upwards on it, and the slice is compressed or squeezed together under the action of these two forces. The internal stress set up in the material of the slice by this compression is, however, able to balance the external stress, and the slice, although slightly compressed, is able to sustain the pressure to which it is subject.

It follows from what has been said that a given piece of matter in the solid state must, under given conditions, have a definite volume and a definite form, and that it is able to offer resistance to any change in configuration which involves a change of volume or a change of form.

In matter in the **liquid state** the molecules are also aggregated together under the control of intermolecular stresses, but the relations between any molecule and those surrounding it are such that it is free to move about in any direction in the liquid. That is, a given molecule in any mass of liquid occupies no particular position in the mass, and offers no resistance to displacement in any direction from any position it may happen to occupy.

This last statement is, however, subject to an important qualification. If force is applied to any molecule in a mass of liquid under such conditions that the molecule can be displaced only by forcing it nearer to, or further from, the surrounding molecules, it offers a very great resistance to displacement. That is, a definite mass of liquid possesses a definite volume, and offers very great resistance to change of volume by compression or expansion.

A mass of liquid * cannot, however, be said to possess a definite form or shape, and it is unable to offer even the smallest resistance to any change of shape which may be impressed on it.

If, therefore, a mass of liquid is subject to the action of any force however small, which tends to change the existing shape of the mass, and is not prevented from doing so by the action of other constraints, the mass will undergo continuous and progressive change of shape, and will extend or flow out in all directions in which it is free to move.

Thus, if we imagine a cube of water, or any similar liquid placed as a cube on a plate, we know from ordinary experience that it would almost instantaneously spread out, or *flow* out into a thin layer covering the bottom of the plate. If we consider what takes place in this case, we can see that any thin horizontal layer of the liquid in the cube is subject to pressure due to the weight of the overlying liquid; this pressure forces the upper and lower molecules of the layer in between the inner molecules in such a way that, while the layer is not in the least compressed, it is compelled to spread out, or *flow* out horizontally in all directions. Since the liquid is not compressed, the molecules offer no resistance to the displacements thus imposed on them, and the flow goes on freely under the action of the smallest force. This process goes on progressively in every layer until the liquid finally comes to rest as a thin layer covering the bottom of the plate.

In the case of a *mobile* liquid, such as water, the process takes place too rapidly to be observed, but in the case of a thick *viscous* liquid, such as syrup, the process is a slow and gradual one, and may easily be observed. If a quantity of syrup, for example, is poured on a large plate, it at first forms an irregularly shaped heap in the middle of the plate. This

* The effects of Surface Tension are not considered here. See Art. 118.

heap, however, gradually spreads out horizontally, and ultimately the liquid flows all over the surface of the plate and comes to rest only when further extension in a horizontal direction is prevented by the sides of the plate.

If a quantity of liquid were poured on a plane horizonal surface of indefinite extent it would spread out horizontally in this way until the film of liquid on the surface is so reduced in thickness that it begins to exhibit effects due to surface tension.

It follows directly from what has been said above, that a liquid may be made to flow or may be *poured* from one vessel to another, and that when poured into any vessel it readily assumes the form imposed on it by the interior of the vessel.

When a mass of liquid is at rest in any containing vessel the conditions are very different to those considered above. Any thin horizontal layer of the liquid is subject to the pressure due to the weight of the overlying liquid, and this pressure, as explained above, tends to make the layer flow outwards in a horizontal direction. This outward flow is, however, prevented by the pressure exerted inwards by the walls of the containing vessel on the edge of the layer. The layer will thus be compressed until the opposing stress set up in the liquid balances the external stress, and equilibrium is established. Every layer of the liquid is in this way supported in equilibrium, and the whole mass of liquid rests in equilibrium in the containing vessel.

Force can be applied to the surface of a liquid only as a pressure or tension* uniformly† distributed over the surface and acting normally or at right angles to the surface; and a mass of liquid can be maintained in equilibrium within any given

* The application of pressure is easily understood; tension can be applied only under certain special conditions, and need not be further considered.

† The weight of the liquid is here neglected.

boundary only by the action of a uniform stress of this nature all over the boundary surface. Thus, if a quantity of liquid, supposed to be without weight, is contained in any vessel open to the air, it will be subject to the atmospheric pressure over its free or exposed surface, and the walls of the vessel will exert everywhere an equal pressure per unit area acting normally over the whole of the surface with which they are in contact.

Since the pressure which can be exerted by any surface on a liquid, or the pressure which a liquid can exert on any surface in contact with it, must, in a weightless liquid, be uniformly distributed over the surface, it is most conveniently measured as the pressure per unit area, for the pressure per unit area must be constant. The pressure exerted on a liquid or by a liquid is, therefore, usually measured and expressed as pressure per unit area.

It will be seen that pressure acting at the surface of a liquid must be at all points normal or at right angles to the surface; if we suppose the pressure acting on the liquid at any point to have a component parallel to, or tangential to the surface, the molecule at that point would be displaced in the direction in which the component acts, for a molecule in a liquid can be displaced in any direction by the smallest possible force. That is, the liquid can be in equilibrium throughout its mass only when the pressure over the surface of the liquid is at all points normal to the surface. It will be seen, too, that the pressure must be uniform if distributed over the boundary surface, for if the pressure per unit area is greater at one point than another, the liquid would flow from the region of highest pressure to the regions of lower pressure. That is, the liquid would flow* through the boundary surface at the areas of low pressure, and equilibrium would be impossible. For example, if we attempt to compress a liquid in a vessel with

* It must be remembered that the liquid is supposed to be without weight.

holes in its walls, the liquid will be forced out through the holes if the external pressure acting on the liquid through the holes is less than that imposed on the liquid by the end-surface of the compressing plunger and the walls of the vessel. If, however, the external pressure over each hole is equal to the internal pressure impressed on the liquid, the mass of liquid in the vessel would be in equilibrium, and no flow would take place through the holes.

It will readily be understood from what has been said that a mass of liquid in equilibrium under the action of a uniform normal pressure at its boundary surface is really subject to this pressure everywhere throughout its mass. That is, if we take, anywhere in the liquid, an imaginary surface separating any two portions of the liquid, the pressure exerted mutually between the two portions across this surface is normal to the surface, and equal to the pressure at the boundary surface.

This pressure expressed, as explained above, as pressure per unit area is, therefore, appropriately called *the pressure in the liquid.*

In the case of a real liquid possessing weight the conditions for the equilibrium of a mass of liquid are complicated by the effect of the weight of the liquid. It will be seen that the pressure impressed on a thin horizontal layer at any depth in the liquid must be greater than the boundary pressure impressed on the free surface of the liquid by the additional pressure due to the weight of the overlying liquid. It follows from this that the pressure at the boundary surface of the liquid cannot be uniform, but must increase with depth below the level of the free surface of the liquid.

The properties of a liquid are more fully considered in Chapter xx.

The plastic state, which occurs in most substances during transition from the solid to the liquid state, is dealt with in Art. 43 in Part iv. on *Heat.*

In matter in **the gaseous state** the molecules are supposed to be so far apart that the intermolecular forces are negligibly small. That is, the molecules are not aggregated together under the control of the intermolecular forces, but are free to move about in any direction quite independently of each other.

It follows from this that a quantity of matter in the gaseous state cannot possess molecular *potential* energy of configuration, for if the intermolecular forces are negligibly small, no work is done *against intermolecular force* in effecting any change of configuration.

It is assumed, in accordance with a theory known as the *kinetic theory of gases*, that the molecules of a gas move about in the space occupied by the gas with great velocity, and that they are constantly in collision with each other, and with the walls of the space in which they are enclosed. The path of any molecule between any two successive collisions is supposed to be a straight line, and although the mean or average length of this free path is really very short, it is long compared with the intermolecular distances in solids and in liquids. The mean length of this free path from collision to collision is called the *mean free path* of the molecule.

If a small quantity of any gas is introduced into any large space unoccupied by any other matter (a vacuum), we know from experience that it at once expands and fills the whole space. Or, if a small quantity of one gas is introduced into a large space already occupied by another gas, we know that it quickly spreads throughout the whole space, for after a very short time indications of its presence may be found in any part of the space.

These results are evidently in accord with the kinetic theory —for by this theory the molecules of a gas are free to extend their excursions in space outwards in all directions, and the only limit which can be set to the space which might be occupied by a given quantity of gas, is the mechanical limit set by the wall or boundary enclosing the space.

It is a well-known experimental fact that a quantity of gas enclosed in any space exerts pressure on the walls of the enclosure. This pressure, which is exerted mutually between the gas and the walls of the enclosure, acts everywhere at right angles to the surface, and is measured by the pressure per unit of area. It is called **the pressure of the gas**, for it is found that it exists everywhere in the gas as a stress acting normally across any interface separating any two contiguous portions of the gas.

The pressure which a given quantity of any particular gas exerts on the walls of the enclosure containing it, is found to depend on the capacity of the enclosure—that is, on the volume occupied by the gas. If this volume is decreased the pressure increases, and if the volume is increased the pressure decreases. That is, if the gas is compressed into a smaller volume the pressure increases, but if it is allowed to expand and occupy a larger volume, the pressure decreases.

These facts are explained on the kinetic theory by supposing that the pressure which a gas exerts on the walls of the enclosure containing it, is due to the continuous bombardment of the walls by the molecules of the gas. Every second a very large, and practically constant, number of molecules moving with high velocities strike and rebound from the walls of the enclosure, and by so doing exert a practically continuous and constant pressure on the walls.

Example.—A rain of small indiarubber balls, each weighing 1 gramme, falls vertically upon a plane horizontal surface, and, on an average, 1,000 balls fall upon every square metre of the surface every second. If the balls strike the surface with a velocity of 20,000 cms. per second, and rebound from it vertically upwards with the same velocity, find the average pressure exerted on each square centimetre of the surface.

Every ball, by its impact with the surface, loses its downward momentum, and gains an equal upward momentum. That is, every ball loses a momentum of (1 × 20,000) C.G.S. units in one direction, and gains (1 × 20,000) C.G.S. units in the opposite direction. The total *change of momentum* which every ball undergoes by its impact on the surface is, therefore, (2 × 20,000) C.G.S. units.

The number of impacts which take place in one second over a square metre of the surface is 1,000, so that the total *change of momentum per second* produced by the resistance offered by a square metre of the surface, to the rain of balls impinging on it, is

$$(1{,}000 \times 2 \times 20{,}000) \text{ C.G.S. units,}$$

or,

$$4 \times 10^7 \text{ C.G.S. units.}$$

But this *rate of change of momentum* measures the resistance offered by the surface in C.G.S. *units of force*. That is, the resistance offered by a square metre of the surface to the rain of balls impinging on it; or, in other words, the pressure exerted by the rain of balls on every square metre of the surface, is 4×10^7 dynes. The average pressure exerted on one square centimetre will, therefore, be 4,000 dynes, or nearly 4·08 gramme-weights.

If this assumption as to the nature of the pressure exerted by a gas on the walls of the enclosure containing it is accepted, it is evident that the pressure must increase as the volume occupied by the gas decreases, for as the volume decreases the molecules become more crowded together, and the number of impacts per second on any given area of the walls must increase. Similarly, the pressure must decrease as the volume occupied by the gas increases, for as the volume increases the molecules become less crowded together, and the number of impacts per second on any given area of the walls must decrease.

It can be shown, too, that the manner in which the pressure of a gas actually varies with its volume, as established by experiment, is the same as the manner in which it ought, theoretically, to vary in accordance with this assumption.

It will be seen from what has been said that the aggregation of the molecules in the gaseous state differs essentially from that which obtains in the solid state or in the liquid state.

A gas resembles a liquid in the *fluidity* which results from its mobility to resist change of form, but it differs essentially from a liquid in its indefinite compressibility and in its power of indefinite expansion.

It should be noticed that resistance is offered by a solid or a liquid both to compression and to expansion, and that this resistance is due in each case to an opposing intermolecular stress set up in the material. In a gas there is no intermolecular stress opposing compression or expansion. Increase of pressure is necessary, as explained above, to produce compression, but expansion takes place freely when the pressure is decreased.

The general properties of a *vapour*, and the distinction between a *vapour* and a *gas*, are dealt with in Chapter ix. of Part iv. on *Heat*. The conditions under which the liquid and gaseous states become continuous, and the *critical state* are also dealt with in the same chapter.

93. **Inertia.**—The inertia of matter, as explained in Art. 35, in dealing with Newton's first law of motion, is one of its most characteristic properties.

It is the property in virtue of which a body—that is, a piece of matter—continues in its state of rest or of uniform motion in a straight line unless acted on by the force.

It will be understood, also, from what has been said in Chapter ix., that it is the property which enables quantity of matter to be measured in units of mass.

94. **Gravitation.**—The power which every piece of matter possesses of attracting every other piece of matter is one of the fundamental properties of matter. The force of attraction exerted mutually between any two pieces of matter is generally known as **gravitation.**

The most familiar example of gravitation is the attraction exerted between the earth and bodies on its surface. The force of attraction exerted by the earth on any body at its surface is usually called *the force of gravity*, and constitutes, as already explained, the *weight* of the body.

Gravitation is not, however, confined to the earth and bodies near it. Every piece of matter in the universe attracts, and is attracted by every other piece of matter in the universe. That is, gravitation is exerted throughout the material universe, and if we wish to emphasise this fact we may use the term **universal gravitation**, instead of the simpler, general term.

The law of gravitation was first correctly stated by Newton in the following form:—

The force of attraction between two particles of matter is directly proportional to the product of their masses, and inversely proportional to the square of the distance between them.

That is, if two particles of masses m and m' are placed at a distance d apart, the force of attraction F, exerted mutually between them, is such that we have

$$F \propto \frac{mm'}{d^2},$$

or

$$F = k \cdot \frac{mm'}{d^2},$$

where k is a constant, known as **the constant of gravitation.**

The formula $F = k\frac{mm'}{d^2}$ applies primarily to the case of two *particles* of matter placed at a distance d apart, for in this case there is no ambiguity as to the distance denoted by d. It can be shown, however, that it applies also to the case of two *spherical* bodies of masses m and m' respectively, placed with *their centres* a distance d apart. It will be understood, too, that the formula may be applied with approximate accuracy in the case of *any* two bodies whose dimensions are small compared with their distance apart.

This law was deduced by Newton from a careful study of the available data relating to the motion of the heavenly bodies. These data had previously been studied by Kepler, and systematised by him into a few general results, known as **Kepler's "Laws."** Newton worked on the assumption that the general law governing the motion of all celestial bodies is the law of gravitation, and he showed that if the law takes the form given above, all Kepler's "laws" can be at once deduced from it, and that the laws stated in this form must, therefore, be in accordance with the data from which Kepler's empirical laws were derived. The truth of the law is now established beyond doubt. It has been since Newton's time, the basis of all astronomical calculations involving the forces acting between bodies moving in space, and the accuracy of the results obtained show that the law must be true.

As an illustration of this Newton showed, by calculation from known data, that the force of attraction exerted by the earth on the moon in accordance with this law is exactly the force necessary to keep the moon moving in its (approximately) circular orbit round the earth with the velocity it actually possesses.

Example.—Show that the acceleration of the moon moving in its circular orbit round the earth is the acceleration due to the force of attraction exerted by the earth on the moon.

The following approximate data will be needed.

Radius of earth, 4,000 miles.

Radius of moon's orbit round the earth is approximately 60 times the radius of the earth.

Time in which the moon makes one complete revolution round the earth is about 27 days 8 hours.

If the velocity of the moon in its orbit round the earth be denoted by v, and the radius of the orbit by r, the acceleration of the moon towards the centre of its orbit—that is, *towards the earth*—is given by $\frac{v^2}{r}$, as explained in Art. 32.

From the data here given the value of v in feet per second is

$$\frac{2\pi \times 60 \times 4{,}000 \times 5{,}280}{656 \times 60 \times 60},$$

and the value of r in feet is

$$60 \times 4{,}000 \times 5{,}280.$$

The value of $\frac{v^2}{r}$ in ft.-per-sec. per sec. is, therefore,

$$\frac{4\pi^2 \times 60 \times 4{,}000 \times 5{,}280}{(656)^2 \times (3{,}600)^2},$$

or ·00897.

That is, the acceleration of the moon in its circular motion round the earth is ·00897 ft.-per-sec. per sec., and is directed towards the earth at the centre of its circular orbit.

The acceleration of a body at the surface of the earth due to its weight—that is, to the attraction of the earth on it—is known to be 32·2 ft.-per-sec. per sec.

The distance of a body from the earth (or from any spherical mass), is its distance from the centre of the earth, so that the distance of the moon from the earth is 60 times the distance of a body at the surface of the earth from the earth. The acceleration of the moon due to the attraction exerted on it by the earth will, therefore, in accordance with the law of gravitation, be $\frac{32 \cdot 2}{(60)^2}$ ft.-per-sec. per sec., or ·00895 ft.-per-sec. per sec., and is directed towards the earth.

The acceleration of the moon towards the earth, calculated from the data of its actual motion, is thus the same (within the limits of the error due to the use of approximate data) as the acceleration to which it is subject, as the result of the attraction exerted on it by the earth calculated in accordance with Newton's law of gravitation.

If we consider the relation—

$$F \propto \frac{mm'}{d^2},$$

which expresses the law of gravitation, it will be seen that the law implies that the attraction between two particles depends only on their masses and their distance apart, and *is quite independent of the material* of which they are made.

That is, in the formula

$$F = k\frac{mm'}{d^2},$$

the gravitation constant k has the same constant value for matter of all kinds, and is not a specific constant having different constant values for different materials.

The truth of this is established by the fact that the acceleration due to gravity is the same for all bodies whatever may be the material of which they are made. Thus, if M denote the mass of the earth, m the mass of a small body at the surface of the earth, and R the radius of the earth, we have

$$F = k\frac{Mm}{R^2},$$

where F is the force of attraction exerted by the earth on the small body. That is, F is the *weight* of the small body and is equal to mg, where g denotes the acceleration due to gravity at the point where the small body is situated. We may, therefore, write—

$$mg = k\frac{Mm}{R^2},$$

or

$$g = \frac{kM}{R^2}.$$

This result shows that if g is the same for all bodies at the same place, k must also be the same for all bodies, for $\frac{M}{R^2}$ is necessarily constant.

Newton and other experimenters investigated the constancy of the acceleration due to gravity for all bodies at the same place by a series of carefully-conducted experiments with the pendulum.

The period of vibration of a simple pendulum has been proved to be given by the relation,

$$t = 2\pi\sqrt{\frac{l}{g}},$$

where l denotes the length of the pendulum, and g the acceleration due to gravity at the place where the pendulum vibrates. From this relation we get—

$$g = \frac{4\pi^2 l}{t^2},$$

so that if g is the same for bodies of all materials at a given place, the period of vibration of a simple pendulum of a given length at that place, should be constant and quite independent of the material of the bob.

It was found, after varying the material of the bob in many different ways, that provided the length of the pendulum, or the length of the equivalent simple pendulum, remained constant, the period of vibration was constant and quite independent of the nature of the material or materials which made up the bob.

Experiment 13.—Make three simple pendulums of exactly the same length by attaching small spherical bobs of lead, brass, and iron, to fine threads about two metres long. Suspend these so that they can vibrate one in front of the other in parallel planes at right angles to the plane in which they hang at rest.

Set the three pendulums vibrating in the same phase with the same amplitude, and note that as long as they continue to vibrate they keep together in the same phase.

This proves that each of the three pendulums has exactly the same period of vibration. That is, the period of vibration is independent of the material of the bob; and it follows from this that the acceleration due to gravity at any place is independent of the material of the body subject to the force of gravity, and also that the gravitation constant is the same for all materials.

The value of the gravitation constant k can be found from the relation—

$$F = k\frac{mm'}{d^2},$$

by determining experimentally the value of F for known values of m, m', and d.

This determination was first made by Henry Cavendish in an historical experiment, now generally known as the **Cavendish experiment.**

The full details of this experiment cannot be given here, but the general method of the experiment must be briefly indicated.

Two small spheres of lead were attached to the ends of a light wooden lever, and the lever was suspended by a long fine wire attached at its middle point so that it hung in a horizontal position. Suspended in this way the lever sets in a definite position of rest, and if deflected from this position in a horizontal plane, the suspension wire becomes twisted, and the moment of the couple due to the torsion on the wire tends to restore the lever to its position of rest. Thus, if T denotes the moment of the couple which is able to twist the wire through unit angle, then the moment of

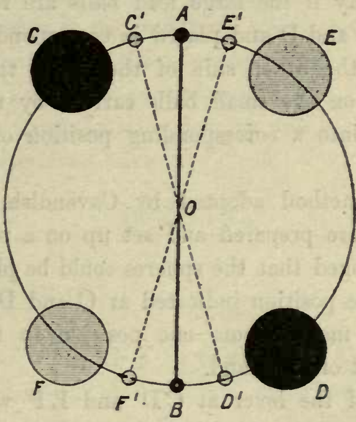

Fig. 110.

the couple due to the torsion of the wire when the lever is deflected through an angle a from its position of rest is Ta in a direction tending to restore the lever to its initial position.

This lever was suspended under conditions suitable to the experiment, and means were provided for observing its position and for measuring accurately any small deflection from that position.

Let AB in Fig. 110 represent the plan of the suspended lever carrying the two small balls of lead at A and B. If now two large spheres or balls of lead are placed at C and D, as shown in the figure, with their centres in the same horizontal plane as the lever, the large balls at C and D will attract the small balls at A and B respectively, and the two forces which thus act on the lever at A and B will tend to deflect it in the same direction from its position of rest. These two forces will in fact constitute a couple which deflects the lever into a new position of equilibrium at C′D′, where the moment of the couple is balanced by the moment of the opposing couple due to the torsion on the suspension wire.

In the same way if the large lead balls are removed from their position at C and D and placed in corresponding positions at E and F on the other side of the lever, the attractions exerted by them on the small balls carried by the lever will deflect the lever into a corresponding position of equilibrium at E′F′.

This was the method adopted by Cavendish. Two large spheres of lead were prepared and set up on a rotating stand which was so arranged that the spheres could be placed, relative to the lever, in the position indicated at C and D or at E and F, and could be moved from one position to the other by a simple movement of the stand.

The positions of the lever at C′D′ and E′F′ were carefully observed, and the angle, C′OE′, between these two positions was measured as accurately as possible.

If this angle, C′OE′, is denoted by $2a$, and if the CD and EF positions of the large spheres are truly symmetrical, relative to the AB position of the lever, the angle through which the lever is deflected by the action of the attraction couple in either of the two positions is denoted by a. Hence, if M denote the mass of a large sphere, m the mass of a small sphere, and d the distance between the centres of the spheres when exerting attraction mutually on each other in a position of equilibrium at C and C′, D and D′, E and E′, and F and F′, the force of attraction between the spheres will be given by

$F = k\frac{Mm}{d^2}$, where k is the constant of gravitation, and in each case this force will act along the line joining the centres of the sphere.

The moment of the attraction couple acting on the lever in either position of equilibrium is given by Fl or $kl\frac{Mm}{d^2}$, where l denotes the length of the arm of the couple. This moment must, however, be equal to the moment of the opposing couple due to the torsion of the wire, and as the twist on the wire is denoted by a when the lever is in either position of equilibrium the moment of this torsion couple will be Ta, where T denotes, as explained above, the moment of the couple able to twist the wire through unit angle.

We must, therefore, have

$$kl\frac{Mm}{d^2} = Ta,$$

or

$$k = \frac{Td^2}{Mml} . a.$$

The value of k can thus be determined, for the values of T, M, m, l, d, and a can all be found experimentally. The value of T is most conveniently found by determining the time of vibration of the lever about its position of rest. It can be shown that if I denote the moment of inertia of the lever and balls round the suspension wire as axis, the time of vibration is given by $t = 2\pi\sqrt{\frac{I}{T}}$, or $T = \frac{4\pi^2 I}{t^2}$.

The value found for k by Cavendish was about $6{\cdot}56 \times 10^{-8}$ in C.G.S. units.

A number of determinations of this constant have been made since the time of Cavendish by different experimenters in England and in other countries. In England a determination was made by Poynting by a method in which the ordinary balance was used instead of the *torsion balance* used by Cavendish. The value of k given by this method in one series of experiments was $6{\cdot}66 \times 10^{-8}$ in C.G.S. units. A later determination was made in 1894 by Boys, who adopted the original torsion balance method, but used a very fine quartz fibre instead of a fine metal wire for the suspension of the lever. The value given by Boys for the constant is $6{\cdot}6576 \times 10^{-8}$ in C.G.S. units.

It will be seen that if the value of k is known it is at once possible to calculate the mass and **the mean density of the earth.** Thus, if m denote the mass of any small body at the surface of the earth at a place where g is the *true* acceleration due to *gravity*, M the mass of the earth, and R its radius, we have,*

$$g = \frac{kM}{R^2},$$

or

$$M = \frac{gR^2}{k},$$

and M can be calculated from this relation for g and R are known.

When M and R are known the mean density of the earth can be found, for we evidently have

$$D = \frac{M}{\frac{4}{3}\pi R^3},$$

or

$$D = \frac{3M}{4\pi R^3},$$

where D denotes the mean or average density of the earth.

If we take $6{\cdot}6576 \times 10^{-8}$, given by Professor Boys, as the value of k, the value obtained for the mean density of the earth is 5·527 grammes per cubic centimetre.

95. **Elasticity.**—Elasticity is a property of matter, in virtue of which a body is able to resist change of size or change

* The form of the earth is assumed to be truly spherical.

of shape, and in virtue of which it tends, while resisting the change, to recover its original size or shape, and is able, when the force causing the change is removed, to recover completely its original size or shape, provided the change has not exceeded certain small limits which differ for different materials.

A body is thus said to offer *elastic resistance* to change of size or change of shape when the stress set up in the material by the change not only opposes the change, but at the same time tends to restore the displaced particles of the body to their original positions. Thus, if a soft ball of clay is squeezed flat between the finger and thumb the resistance it offers to the change of shape is not elastic resistance, for although the friction between the particles of clay opposes the displacement of one particle relative to another, it does not at any stage in the process tend to restore the displaced particles to their original positions. The resistance in this case is *frictional* in character, and resembles the resistance offered by a viscous liquid to change of shape, as explained below.

In general a body is able to offer elastic resistance to change of size or change of form, only within certain narrow limits of change. These limits for any material are called *the limits of electricity* for that material, and are found to differ considerably for different materials.

The substances of highest elasticity are those which, like steel, glass, ivory, and most liquids, offer very great resistance to change of size or change of shape. Certain substances, such as indiarubber, are commonly called *elastic* substances, because the limits of elasticity for these substance are unusually wide. A piece of indiarubber, for example, offers elastic resistance to change of size and shape through a very wide range of change, and is able to recover its original size and shape after undergoing very large changes of this kind.

A solid substance is able to offer elastic resistance to change of size, and also to change of shape. That is, a solid substance possesses elasticity of bulk or elasticity of volume, as well as elasticity of shape or form.

A liquid substance, on the other hand, is able to offer elastic resistance only to change of size, and offers no elastic resistance whatever to change of shape. That is, a liquid possesses only elasticity of volume, and is devoid of elasticity of form. This, in fact, constitutes the essential difference between a solid and a liquid: a solid possesses elasticity of form in a very marked degree, but a liquid has no trace of this property.

A gas like a liquid has elasticity of volume, but no elasticity of form.

When force is applied to a body in order to produce change of volume or change of form, it is generally assumed to be applied as a pressure or a tension exerted uniformly over the whole surface of the body, or over a portion of the surface, and is supposed to act either normally or tangentially to this surface. The force applied is, therefore, generally measured as *the force per unit of area*, and when so measured is known as the **stress** to which the change it produces is due.

The change of size produced in any body by a suitable stress is not measured by the actual change of volume produced, but by *the ratio of this change to the initial volume*. Similarly, change of shape is measured, as explained below in Art. 98, by the ratio of a linear displacement to a length definitely associated with the displacement.

Change of size or change of shape, measured in this way as a ratio or a proportional change, is called a **strain**, and any body in which a change of this kind is produced is said to be *strained.*

When a body is strained within the limits of elasticity for the material of which it is made, it is found by experiment that the strain produced is directly proportional to the stress applied. That is, for small strains within the elastic limits of the material considered,

$$\text{the ratio } \frac{\text{stress}}{\text{strain}} \text{ is constant.}$$

This constant ratio for any material is the **modulus of elasticity** for the particular case of stress and strain to which it applies.

96. **Density and Specific Gravity.**—The mass of any volume of a given uniform material* is obviously proportional to the volume. Thus, the mass of n cub. cms. of pure water at 0° C is n times the mass of 1 cub. cm. of pure water at the same temperature.

It follows from this that the mass per unit volume for any definitely specified material is constant for that material. Further, if the mass per unit volume for different materials is compared it is found that, although it is constant for a given material, it differs widely for different materials. Thus, the mass of 1 cub. cm. of gold is about 19·3 grammes, the mass of 1 cub. cm. of silver is about 10·5 grammes, the mass of 1 cub. cm. of copper is about 8·9 grammes; while the mass of 1 cub. cm. of pure water at 4° C. is almost exactly 1 gramme.

The mass per unit volume of any substance is thus a characteristic or specific constant of the substance, and is called the **density** of the substance.

Hence, if the mass of a body of any uniform material is denoted by m, and its volume by v, the density, d, of the material is given by the relation

$$d = \frac{m}{v}.$$

This formula may be written in the form, $m = vd$, and establishes a very important relation between mass, volume, and density, for any uniform material.

When a body is not of uniform material throughout, its

* The material is here supposed to be uniform with regard to the distribution of its mass throughout its volume.

density is not uniform, but varies from point to point, and is measured at any point by the mass per unit volume for a very small volume of the material taken at that point. This case need not, however, be further considered.

It will be seen from what has been said that the experimental determination of the density of any material involves the measurement of the mass and volume of a selected portion of the material. Thus, to find the density of silver by a *direct* experimental method it would be necessary to take a suitable piece of silver, and to find its mass by weighing it, and its volume by measuring it. Then, if m and v denote respectively the mass and volume thus determined, the density of silver could be calculated from the relation

$$d = \frac{m}{v}.$$

It is explained below, however, that although this direct method may be adopted, and is adopted in the case of gases, it does not give such accurate results as the indirect methods, dealt with in Chap. xix. The inaccuracy of the method depends upon the fact that, although the mass of a body can be determined by weighing with the highest accuracy, the volume of a body cannot be determined by *direct measurement* with anything like the same degree of accuracy.

Density, as defined above, is sometimes called *absolute density.* Instead of expressing the density of a substance *absolutely*, as explained above, it may evidently be expressed *relatively* with reference to the density of some well-defined substance as a standard. The density of a substance, expressed relatively to the density of a specified standard substance, is called the *relative density* of the substance. The standard substance selected for reference is pure water at 4° C.

It will be seen that the relative density of a substance is thus the ratio of the density of the substance to the density of water at 4° C., and is merely a number expressed without units.

Thus, the absolute density of gold is 19·3 grammes per cub. cm., and the absolute density of water at 4° C. is 1 gramme per cub. cm., so that the relative density of gold is 19·3.

It is one of the advantages of the C.G.S. system that the relative density of a substance is expressed by the number which measures its absolute density. This is not the case in the English system. For example, the absolute density of gold in English units is about 1207 pounds per cubic foot, and the absolute density of water is about 62·5 pounds per cubic foot, so that the relative density of gold is $\frac{1207}{62 \cdot 5}$ or 19·3. It is important, however, to note that the relative density of a material must be the same in all systems of units.

The **specific gravity** of a substance is essentially the same at its relative density, and may be defined as the ratio of the weight of any volume of the substance to the weight of the same volume of water at 4° C.

Thus, if W′ denote the weight of *any volume* of a given substance, and W the weight of the *same volume* of water at 4° C., then

$$\frac{W'}{W} = s,$$

where s denotes the specific gravity of the substance. It will be seen here that if V denote the volume of the substance, d' its density, and d the density of water at 4° C., we have $W' = Vd'$, and $W = Vd$, so that

$$\frac{W'}{W} = \frac{Vd'}{Vd} = \frac{d'}{d}.$$

That is, $s = \frac{d'}{d}$, or the specific gravity, s, of any substance is essentially the ratio of the density of any substance to the density of water at 4° C., and is, therefore, the same as the relative density of the substance.

The relative densities or specific gravities of a few of the commoner substances are given in the following table.

Table of Relative Densities or Specific Gravities.

Solids.

Aluminium	2·6 – 2·8	Carbon (graphite)	1·9 – 2·2
Copper	8·8 – 8·95	Carbon (diamond)	3·5 – 3·6
Gold	19·25 – 19·35	Sulphur	2·0
Iron (wrought)	7·8 – 7·9	Brick	2·0 – 2·2
Iron (cast)	7·1 – 7·7	Chalk	1·9 – 2·8
Steel	7·8 – 7·9	Coal	1·2 – 1·8
Lead	11·35	Flint	2·65
Nickel	8·3 – 8·8	Granite	2·4 – 3·1
Platinum	21·3 – 21·6	Glass	2·4 – 2·8
Silver	10·4 – 10·6	Marble	2·6 – 2·8
Tin	7·3	Porcelain	2·4 – 2·6
Zinc	7·1 – 7·2	Quartz	2·65
		Slate	2·6 – 2·7
Brass	8·4 – 8·7	Pitch	1·1
Bronze	8·7 – 8·9	Paraffin wax	·88 – ·92
German silver	8·3 – 8·5	Sand	1·5 – 1·7

Ash	·7 – ·9	Ebonite	1·1 – 1·2
Oak	·6 – ·9	Indiarubber	·9 – 1·0
Beach	·7 – ·9	Ivory	1·8 – 1·95
Box	·9 – 1·1	Bone	1·5 – 2·0
Deal	·4 – ·6	Sugar	1·6
Willow	·4 – ·6	Rock salt	2·3 – 2·4

Liquids.

Alcohol	·79	Petroleum	·8 – ·9
,, (methyl)	·81	Sea water	1·025
Benzene	·90		
Chloroform	1·48		
Ether	·74	Sulphuric acid	1·85
Glycerine	1·26	Nitric acid	1·56
Linseed oil	·94	Hydrochloric acid	1·27
Olive oil	·92		

Water at 4° C.	1
Mercury at 0° C.	13·596

The density of **gases** is considered in Art. 126. The absolute density of **dry air** at 0° C., and under normal atmospheric pressure, is 1·293 grammes per litre, or ·001293 gramme per cubic centimetre.

The density of a substance generally decreases as the temperature rises,* for as the substance expands with rise of temperature, the volume occupied by any given mass must necessarily increase, and the density must, therefore, decrease.

The variation of density with change of temperature has been very carefully studied † experimentally for water and mercury. A short tabular statement of the results obtained for temperatures within the ordinary range is given below.

Water.

The absolute density of pure water free from air, and under normal pressure at 4° C., is found to be ·999955 gramme per cubic centimetre.

Specific Gravity of Water at Different Temperatures Relative to Water at 4° C.

Temperature.	Specific Gravity.	Temperature.	Specific Gravity.
0° C.	·999868	16° C.	·998970
2	·999968	18	·998622
4	1·000000	20	·998230
6	·999968	22	·997796
8	·999876	24	·997322
10	·999727	26	·996810
12	·999525	28	·996259
14	·999271	30	·995672

Mercury.

The density of mercury at 0° C. is usually taken as 13·596 grammes per cubic centimetre.

* See Art. 17 in Part iv. on *Heat*.

† See Art. 28 in Part iv. on *Heat*.

Density of mercury at different temperatures calculated from Regnault's value of the coefficient of cubical expansion of mercury.

Temperature.	Density.	Temperature.	Density.
0° C.	13·596	15° C.	13·559
5	13·584	20	13·547
10	13·571	25	13·534

CHAPTER XVII.

PROPERTIES OF SOLIDS.

97. **Volume Elasticity.**—When a body is strained in such a way that it undergoes change of volume *without change of shape*, the elasticity which enables it to resist the change is known as **volume elasticity** or **bulk elasticity.**

A body can be strained in this way only by subjecting the body to a uniform pressure acting normally all over its surface, and then increasing or decreasing this pressure according as it is required to increase or decrease the volume of the body. Pressure may conveniently be applied in this way by immersing the body in a suitable liquid and then applying pressure to the liquid. The pressure on the liquid is exerted on the surface of the immersed body, and acts everywhere at right angles to the surface. The *piezometer* apparatus, described in Art. 116, has been used for applying pressure to solid bodies in this way. It is, however, difficult to obtain in any way reliable measurements of the compressibility of a solid: most solids offer very great resistance to compression, so that the changes of volume, even under very great changes of pressure, are very small, and in any piezometer method they are subject to correction for the compression of the liquid used.

By using a compressible substance, such as cork, the meaning of change of volume, without change of shape, may be illustrated in a striking manner by means of the piezometer apparatus. If a ball of cork of uniform structure is subjected to pressure, it may be compressed to a ball of much smaller volume without losing its spherical form, and when the pressure is removed it recovers its original volume, retaining its spherical form throughout the process.

A body may be subjected to very great hydrostatic pressure by sinking it to a great depth in the sea. A piece of cork may in this way be made so dense by compression that it sinks.

Let a body occupy a volume V when subjected to a uniform pressure of P units per unit area, and let the volume be decreased to $(V-v)$, without change of form, when the pressure is increased to $(P+p)$ units per unit of area.

The **stress** to which the change of volume in this case is due is measured by p, and the **volume strain** produced is given by the ratio $\frac{v}{V}$. The **modulus of volume elasticity**, usually denoted by k, is given therefore by

$$k=\frac{p}{v/V}=\frac{pV}{v}.$$

As explained above, it is very difficult to measure v in this relation with any accurracy, so that the value of k cannot be determined accurately by any direct experimental method. It can, however, be determined *indirectly*, as explained below.

98. **Simple Rigidity.**—The elasticity which enables a solid body to resist change of shape or form, without change of volume, is known as **form elasticity**, or, more generally, as **simple rigidity**.

The fundamental strain in any change of shape without change of volume is the displacement in its own plane of any very thin plane layer of the material, relative to an adjacent parallel layer. Thus, if the thin plane layer AB, in Fig. 111, is displaced in its own plane through a distance AA′ or BB′, relative to the adjacent parallel layer CD, the material made up of the two layers is subject to a simple *form strain*, without attendant change of volume, and the strain is measured by the ratio of the displacement, AA′, to AC, the perpendicular distance between the layers.

Similarly, if a plane layer AB, Fig. 112, is displaced in its own plane (through a distance d) relative to a plane parallel layer CD, at a perpendicular distance D from it, the material between the layers is subject to simple change of form without change of volume, and the *form strain* thus produced is measured by the ratio $\frac{d}{D}$.

This form of strain is known as a **shearing strain**, or a **simple shear.** The forces by which this strain is produced must evidently be applied *tangentially* over the surfaces of the plane layers AB and CD, and must act on these layers in opposite

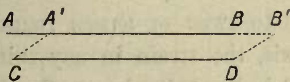

Fig. 111.

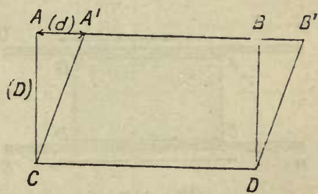

Fig. 112.

directions in the line of displacement. The force per unit area thus applied to either surface is taken as the measure of the applied **stress,** and the ratio of this **stress** to the strain d/D is called the **modulus of simple rigidity** for the material under strain. This modulus is usually denoted by n, so that if P denote the tangential stress and s the shearing strain then

$$n = \frac{P}{s}.$$

It can easily be seen that a shearing strain involves no change of volume. Thus, if a rectangular block of any material be sheared by the relative displacement of any two parallel faces, the volume of the block must remain unchanged; for, if we consider the block made up, like a pack of cards glued together, of a very large number of thin layers parallel to the two faces considered, any one layer is simply displaced slightly *in its own plane*, relative to the adjacent layers, and the length, breadth, and depth of the block must remain unchanged.

A block of any material might conceivably be sheared by applying to any two parallel faces uniform stresses acting in opposite directions and parallel to the surface in each case. Thus, in Fig. 113, if we suppose the block ABCD to have its upper and lower faces glued to the planes PQ and RS, the block could be sheared by applying equal forces to displace the planes PQ and RS, each in its own plane, in opposite directions, as indicated by the arrows in the figure.

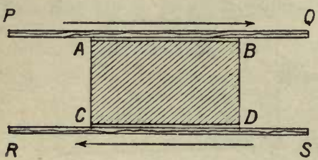

Fig. 113.

It can be shown, however, that when a cylinder is subjected to twist or *torsion* round its axis, the strain in any thin cylindrical shell is really a shearing strain which increases as the radius of the shell increases. Thus, let ABCD, in Fig. 114, represent a cylinder, and suppose the upper face of this cylinder to remain fixed, while the lower face is twisted through a small angle α relative to the upper face in the direction of the arrow.

If we consider the effect of this twist on the outermost shell of the cylinder, it will be seen that any strip of the shell, such as *abcd*, taken parallel to the axis before twisting, is displaced by the twist into the position *abc'd.'* That is, the strip is subjected to a simple shear, and the shearing strain produced in it measured by the ratio $\frac{cc'}{ac}$. From this result it will be seen that the shearing strain in any cylindrical shell of radius x, when the lower face of the cylinder is twisted in its own plane through an angle a relative to the upper face, is measured by the ratio $\frac{xa}{l}$, where l denotes the length of the cylinder. Since

$$\frac{\textbf{stress}}{\textbf{strain}} = n,$$

it follows in this case that, **stress** $= n$. **strain** $= \frac{nxa}{l}$. That is, the

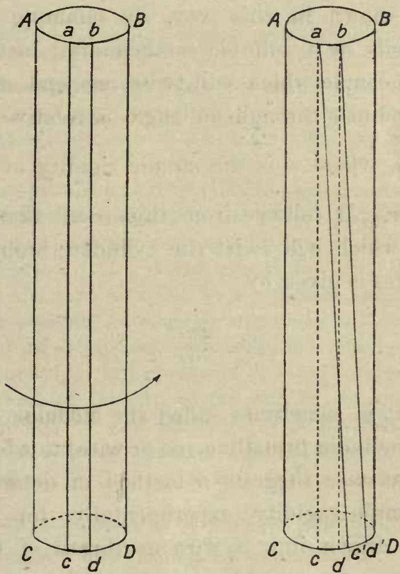

Fig. 114

stress or force per unit area applied to the lower face of the shell of radius x in order to twist it through an angle a relative to the upper face is $\frac{nxa}{l}$, where l is the length of the cylinder. This force is applied uniformly over the surface and acts everywhere parallel to the surface and tangential to the section of the shell, so that if δ denote the thickness, radially, of this shell, *the moment of the force causing the torsion of the shell* is $2\pi x\delta \cdot \frac{nxa}{l} \cdot x$, or $\frac{2\pi n x^3 \delta a}{l}$. In the same way the moment of the force causing the torsion in every one of the thin cylindrical shells, into which the cylinder may be supposed to be divided, can be determined, and by taking the sum of the moments for all the shells we can get the value of the moment of the couple which produces the torsion of the cylinder as a whole.

It can be shown in this way, by summing the moments for all the shells by a suitable mathematical method, that the moment of the couple which will twist one end of a cylinder of length l and radius r through an angle a, relative to the other end, is $\frac{\pi n r^4}{2l} a$, where n is the simple rigidity of the material of the cylinder. It follows from this result that the moment of the couple which will twist the cylinder through unit angle (*circular measure*) is given by

$$T = \frac{\pi n r^4}{2l}.$$

This moment is sometimes called the **modulus of torsion** for the particular cylinder (usually a rod or wire) to which it applies.

This result at once suggests a method of determining n, the modulus of simple ragidity, experimentally, for any material which can be drawn into a wire or thread of truly circular section.

A heavy bob, in the form of a metal sphere or cylinder, is attached to one end of a convenient length of fine wire made of the material for which the modulus of simple rigidity is to be determined and then suspended, as shown in Fig. 115, by fixing the other end of the wire in a suitable clamp. If the bob used is cylindrical in form it is best to attach it to the wire, so that it hangs (as shown in the figure) with its axis in the same vertical line as the axis of the wire. The arrangement thus set up is called a *torsion pendulum*, and may evidently be set in vibration under the influence of the torsion of the wire by twisting the bob round its axis through any convenient angle, and then letting it go.

If the vibrations of a torsion pendulum are studied experimentally, it will be found that the period of vibration is perfectly constant, and quite independent of the angular amplitude of the vibrations, whether this amplitude be large or small.*

Experiment 14.—Set up a torsion pendulum similar to that shown in Fig. 115. Attach a very light wire pointer horizontally to the lower end of the bob, and arrange a circular scale immediately below, so that the pointer indicates the zero of the scale when the bob is at rest.

Now determine the period of vibration under torsion for different amplitudes. The amplitude may be varied from two or three complete revolutions down to a few degrees. The period should be found by determining the time occupied by a sufficient number (as many as possible) of complete vibrations within a certain range of amplitude. The complete vibrations should be counted, as in Experiment 3, as the intervals between successive transits of the bob in the same direction through its zero position. This position is easily observed by means of the pointer and scale.

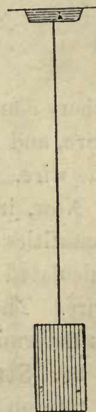

Fig. 115.

It will be found that the period is quite constant for all amplitudes.

The period of vibration of a torsion pendulum can thus be easily and accurately determined by experiment.

It can be shown, however, that the period of vibration of the pendulum is given by

$$t = 2\pi\sqrt{\frac{I}{T}},$$

where T is the modulus of torsion of the wire and I the

* This result shows, incidentally, that the moment of the couple due to the torsion of the wire is directly proportional to the angle of the twist. This is in accordance with the theoretical result obtained above.

moment of inertia of the vibrating system round the axis of rotation.

From this relation we get

$$T = \frac{4\pi^2 I}{t^2}.$$

That is,

$$\frac{\pi n r^4}{2l} = \frac{4\pi^2 I}{t^2},$$

or

$$n = \frac{8\pi I l}{r^4 t^2},$$

where r and l denote respectively the radius and length of the wire, and n the modulus of simple rigidity for the material of the wire.

Now, in any experiment, such as that described above, the quantities t, r, and l can be measured directly, and I can be calculated from mass, dimensions, and form of the bob and wire. The value of n for the material of the wire can thus be determined with considerable accuracy by direct experiment.

99. **Stretching.**—When a wire or rod is stretched by a tension in the direction of its length the strain produced is not a pure *volume strain* or a pure *form strain*, but involves a change both of volume and of form. The elastic resistance which a material offers to stretching involves, therefore, both the volume elasticity and the simple rigidity of the material.

It can readily be shown that stretching a rod produces a true change of form in the material of the rod. When any portion of the rod is stretched its length increases, but the dimensions at right angles to the length decrease. Hence, if we consider a spherical portion of the material of the unstretched rod, taken in the interior of the rod as at A in Fig. 116, it will be seen that this spherical portion must, in the stretched rod, take the ellipsoidal form indicated at B in the figure.

Stretching is thus accompanied by true change of form. It is accompanied also by change of volume for experiment shows, as explained below, that a rod or wire when stretched increases in volume.

When a wire of length L is elongated by stretching by an amount l, the elongation of each unit length of the wire is obviously the same, and is measured by the ratio $\frac{l}{L}$. This ratio which may be defined as the *elongation per unit length*, or the ratio of the total elongation to the initial length of the wire, is taken as the *strain* due to stretching.

Also, if W denote the stretching force or tension applied to the wire in stretching it, and a denote the area of cross section of the wire, the *stress* to which the stretching is due is evidently given by $\frac{W}{a}$, for the tension is necessarily distributed uniformly over the cross section.

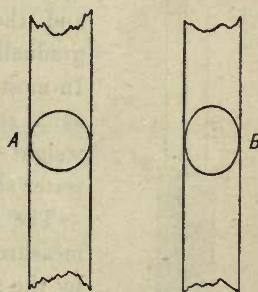

Fig. 116.

Experiment shows that for all stretching strains within the limits of elasticity of the material, the ratio of stress to strain, where each is measured as explained above, is constant. This ratio may, therefore, be considered as the modulus of elasticity for stretching, and is known as **Young's modulus of stretching**, or simply as **Young's modulus.** Hence, if Young's modulus for the material of the wire considered above is denoted by M, we have

$$M = \frac{\frac{W}{a}}{\frac{l}{L}} = \frac{WL}{al}.$$

The value of M for the material of any wire or rod can thus be determined experimentally by stretching the wire under such conditions that W, L, a, and l can be accurately measured, and then calculating out the value from the relation given above.

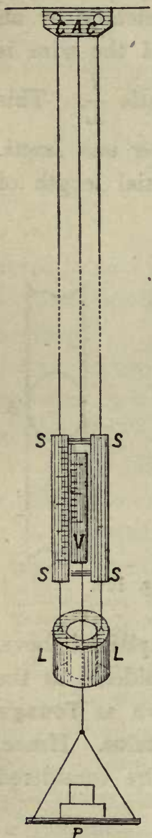

Fig. 117.

A convenient form of apparatus for carrying out a determination of Young's modulus by this method is shown in Fig. 117.

The wire to be stretched is suspended, as shown at A in Fig. 117, from a vice clamp CC in which its upper end is fixed, and carries at its lower end a vernier V and a scale pan P, as shown in the figure. The free length of the wire for stretching should be about 2 metres, and the stretching force should be applied gradually by placing weights in the scale pan. In most cases it is convenient to use a bucket as a scale pan, and to apply the stretching weight by pouring measured quantities of water slowly into the bucket.

The scale, SS, on which the elongation is measured is carried from the suspension clamp by the two side wires shown in the figure, and is held in a fixed position relative to the vernier by means of the heavy tubular weight of lead shown at LL. When weights are placed in the pan so as to stretch the wire, the vernier V is pulled downwards in the groove in which it moves freely relative to the scale, and the difference in the vernier readings before and after the addition of any weight gives the elongation due to that weight. The elongation produced by any weight can thus be measured with considerable accuracy by using a good scale and an accurately divided vernier reading to a small fraction of a scale division.

The initial length of the wire, between the clamp and the vernier, can be measured with sufficient accuracy by direct application of a metre scale; and the area of cross section can be found by taking a number of careful measurements of the diameter with a screw gauge, and then calculating the area from the mean value of the diameter.

The stretching weight to which the measured elongation is due is given by the weights placed in the scale pan. All the data necessary for the determination of Young's modulus from the relation,

$$M = \frac{WL}{al},$$

can thus be found by direct experiment.

Example.—In an experiment for determining Young's modulus for copper by stretching a copper wire, the following data were obtained:—

Initial length of wire, 206·2 cms.
Mean diameter of wire, ·0914 cm.

Load on Wire. (In addition to starting load necessary to straighten wire.) Kilogramme-weights.	Elongation due to Additional Load. (Cms.)	Elongation per Kilogramme-weight. (Cms.)
5	·155	·0310
10	·320	·0320
15	·470	·0313
20	·635	·0317
Average elongation per kilogramme-weight,		·0315

Substituting in the relation,

$$M = \frac{WL}{al},$$

from these data we have

$$W = 1 \text{ kilogramme-weight}$$
$$= 1{,}000 \times 981 \text{ dynes.}$$
$$a = \cdot 00657 \text{ sq. cm.}$$
$$l = \cdot 0315 \text{ cm.}$$
$$L = 206 \cdot 2 \text{ cms.}$$

We, therefore, have

$$M = \frac{1{,}000 \times 981 \times 206 \cdot 2}{\cdot 00657 \times \cdot 0315} \text{ dynes per sq. cm.}$$
$$= 9 \cdot 77 \times 10^{11} \text{ dynes per sq. cm.}$$

That is, Young's modulus for copper is given by this experiment as $9 \cdot 77 \times 10^{11}$ in dynes per square centimetre.

The behaviour of a wire when stretched beyond the limits of elasticity for the material illustrates the general behaviour of a material when strained in any way beyond the limits of perfect recovery from the strain.

The data obtained by stretching a wire beyond the elastic limit serve also to determine the *breaking stress* at which the wire breaks under tension. This breaking stress is taken as a measure of the tenacity or tensile strength of the material of the wire.

The breaking stress for a thin wire may be determined experimentally by an extension of the method, described above, for the determination of Young's modulus. The apparatus shown in Fig. 117 may be used, but a very long scale suitable for measuring considerable elongations should be used. The wire is stretched by gradually increasing the load in the scale pan, but when the elastic limit is reached the increments of load must be made smaller and smaller, and must be added very gradually —preferably by pouring water or shot into a bucket used as a scale pan. This process of gradual stretching is continued until the wire breaks.

If the data obtained in such an experiment are examined it will be found that the elongation produced by a given increment of load is practically constant up to the load at which the elastic limit for stretching is reached. Beyond this limit, however, the elongation for a given increment of load steadily increases until a point known as the *yield point* of the material is reached. At this point the wire "yields" or "gives" under the applied load, and a considerable permanent elongation is produced without any further addition to the load being made. Beyond this point a stage is reached in which the load must again be increased in order to obtain further elongation, and in this stage, as in that immediately preceding the yield point, the elongation for a given increment of load rapidly increases until ultimately the *breaking point* of the wire is reached. During this stage the material of the wire appears to be in a more or less plastic condition, for it is found that the elongation due to a given increment of load increases with the time for which the load is applied. It is found, too, that as the breaking point is approached the wire is drawn out more at some points than others, and its cross section ceases to be of uniform area.

If a curve is plotted showing how the elongation of the wire increases with the stretching stress, it will have the form shown in Fig. 118. The straight portion OA of the curve shown in the figure represents the relation between the elongation and the stress within the limits of elasticity where the elongation is directly proportional to the stress. At the point A, where the continuation of the straight portion OA leaves the experimental curve, the elastic limit is reached, and the limiting stress at which this takes place is represented by OP.

The portion AB represents the relation between the elongation and the stress between the elastic limit at A and the yield point at B, and it can be seen from the form of the curve that in this portion of it, the elongation due to any given increment of load, rapidly increases as the load increases, until the point B is reached, where a considerable elongation takes place under the *constant* load represented by OQ. The portion BC represents the relation between the elongation and the stress between the yield point at B and the breaking point at C, and in this portion, too, it will be seen that the elongation increases very rapidly with the load until the breaking point is reached. The stretching stress corresponding to this breaking point, represented in the figure by OR, is the breaking stress for the material of the wire. It must be noted, however, that the breaking of the wire is often due to some small flaw in

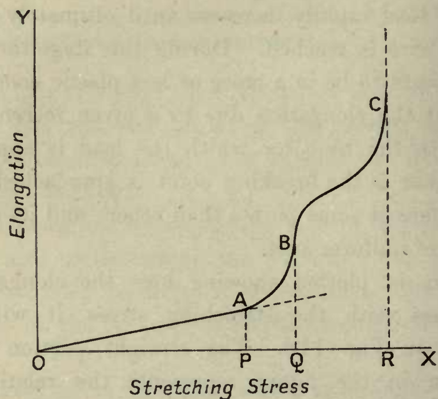

Fig. 118.

the wire, and may, therefore, take place with the same wire under widely different stresses. It is necessary, therefore, in determining the breaking stress to eliminate accidental breakages, and to find the maximum *constant* stress at which breakage takes place.

It will readily be understood that the whole process of stretching a wire, as represented by the curve OABC in Fig. 118, can be followed out experimentally only when the material of the wire is of a very *ductile* and tenacious character. In the case of *brittle* material the stages represented by the portion ABC of the curve are practically non-existent, and the breaking point is reached very quickly after the elastic limit is passed.

Experiment 15.—Find the breaking stress for soft iron by the method indicated above. Use about a metre length of No. 22 wire.

A serious difficulty which presents itself in all experiments dealing with strains beyond the limits of electricity for the strained material, is the effect of time on the strain due to any given stress. In an experiment, such as that described above, it is found, even before the yield point is reached, that the elongation produced by any load beyond the elastic limit is not constant, but increases slightly with the time for which the load is allowed to act.

The *experimental laws of stretching* may be expressed concisely by saying that, within the limits of elasticity the stretching stress and the stretching strain, as defined above, are directly proportional the one to the other. They may be expressed more formally from the relation—

$$M = \frac{WL}{al},$$

which gives

$$l = \frac{1}{M} \cdot \frac{WL}{a}.$$

That is, the elongation of a wire of given material due to stretching within the limits of elasticity for the material, is directly proportional to the stretching force, directly proportional to the initial length of the wire, and inversely proportional to the area of cross section.

It should be noted that compression *in one direction* is essentially the same strain as stetching in one direction.

That is, whether a rod is stretched or compressed in the direction of its length the modulus of elasticity involved in either case is Young's modulus of stretching.

Young's modulus is, like the modulus of simple rigidity, a very important elastic constant for any material. These two moduli are the moduli which can be determined most accurately by *direct experiment*, and as other moduli of elasticity, such as the modulus of volume elasticity, can be derived from them *theoretically*, they serve to determine *indirectly* the values of these moduli.

100. **Bending.**—When a rod or beam is bent in any way, as shown in Fig. 119, the layers on the convex side are obviously stretched, while those on the concave side are compressed in the direction of their length. A certain neutral layer, indicated in outline in the figure, is neither stretched nor compressed, but all layers on the concave side of this layer are compressed, and all on the convex side are stretched.

Fig. 119.

The modulus of elasticity involved in bending is, therefore, Young's modulus of stretching, and this modulus may be determined experimentally by data derived from the bending of a beam.

The theory of bending strains is too complicated to consider here, but the experimental laws of bending for light beams of rectangular section loaded at one point, may be established by a few simple and interesting experiments, such as those indicated below. For the purpose of these experiments a beam may be bent by fixing it securely in a horizontal position with one end firmly clamped and the other end free, and then applying a weight at the free end, as in Fig. 120, or by supporting it in a horizontal position on two knife-edges, placed one under each end, and then applying the weight at a point midway between the supporting edges as shown in Fig. 121. In either case the bending is conveniently measured by the downward deflection of the point at which the weight is applied.

The bending deflection produced in a light beam of rectangular section of given material when loaded at a particular point, must evidently depend upon (i) the load applied; (ii) the length of the beam; (iii) the width of the beam; and (iv) the depth or thickness of the beam. It is necessary, therefore, to find by experiment how the deflection depends upon each of these four factors separately when the other three are kept constant.

Experiment 16.—Find how the bending deflection of a light beam loaded at its middle point depends upon the **load** to which the bending is due.

Set up a light wooden beam about a metre long and about 2 cms. × 1 cm. in cross-section, in the manner indicated in Fig. 121, and

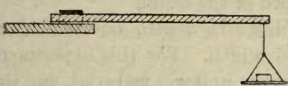

Fig. 120.

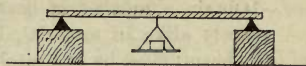

Fig. 121.

arrange a scale and pointer to measure as accurately as possible the downward deflection of the middle point of the beam.

Measure carefully the deflections produced by a series of gradually increasing loads, taking care not to pass the elastic limit for the beam, and then arrange the results in a tabular form similar to that given in the Example in Art. 99 above.

It will be found that the bending deflection is directly proportional to the load, and that the average deflection per unit load is, therefore, constant for all loads within the elastic limit.

Note that in this experiment the load only is varied.

Experiment 17.—Find how the bending deflection of a light beam loaded at its middle point depends upon the **length** of the beam.

Set up a wooden beam as in the foregoing experiment, and find the deflections produced by the same load for different lengths of the same beam. The length of the beam subjected to bending is very easily varied by simply varying the distance between the knife-edges which support the beam.

It will be found, on tabulating and reducing the results obtained, that the bending deflection is directly proportional to the cube of the length.

Note that in this experiment the length of the beam only is changed.

Experiment 18.—Find how the bending deflection of a light beam loaded at its middle point depends upon the **width** of the beam.

Obtain a number of beams cut from the same piece of wood of the same length and thickness, but of different widths. Set up each beam in turn, as in the foregoing experiments, and measure the deflection produced by the same load for the same length of beam in each case.

It will be found on tabulating and reducing the results that the bending deflection is inversely proportional to the width of the beam.

This result might have been deduced on first principles without appealing to experiment.

Note that in this experiment the *width* only is varied.

Experiment 19.—Find how the bending of a light beam loaded at the middle point depends upon the **depth** of the beam.

Obtain a number of beams which differ in depth, but which are exactly alike in material, length, and width. For this experiment the beams must be accurately made of very uniform material, and the depth should vary within comparatively narrow limits. Five beams, varying in depth from 8 mm. to 10 mm., in steps of half a millimetre, would answer the purpose.

Set up each beam in turn, and find the deflection of the same length under the same load.

It will be found, on tabulating and reducing the results, that the bending deflection is inversely proportional to the cube of the depth or thickness of the beam.

Note that in this experiment the **depth** only is varied.

If the foregoing experiments are repeated for a beam loaded at one end, as in Fig. 120, exactly similar results will be obtained.

It will be seen from the results of these experiments that the experimental laws of bending for a light beam of rectangular section loaded at its middle-point or at one end may be stated as follows:—

The deflection of the point at which the load is applied is directly proportional to the load, directly proportional to the *cube* of the length, inversely proportional to the width, and inversely proportional to the *cube* of the depth or thickness of the beam. That is, if l denote the deflection, W the load, L the length, a the width, and b the depth of the beam, we have

$$l \propto \frac{WL^3}{ab^3},$$

or

$$l = K\frac{WL^3}{ab^3},$$

where K is a constant.

The value of this constant depends upon the value of Young's modulus for the material of the beam. It can be shown that its value is $\frac{4}{M}$ when the beam is loaded at the middle, and $\frac{1}{4M}$ when the beam is loaded at one end.

The value of Young's modulus for the material of a beam can thus be determined from the relation given above. In the case of a beam loaded at the middle, for example, we have

$$l = \frac{4}{M} \cdot \frac{WL^3}{ab^3},$$

or

$$M = \frac{4WL^3}{ab^3 l}.$$

Hence, if in any experiments, such as those outlined above, the values of W, L, a, b, and l are carefully and accurately measured, the value of M can be at once obtained by means of this formula.

Experiment 20.—Find Young's modulus for glass by the method of bending. Use a fairly long strip of plate glass, of uniform width. The bending deflection may conveniently be measured by a spherometer or screw gauge suitably arranged for the purpose.

101. **Torsion.**—The theory of torsion in the case of a cylindrical rod or wire has already been considered in Art. 98, as far as it can be dealt with here.

It has been shown that a twisting strain is essentially a shearing strain, and that the modulus of elasticity involved in torsion is, therefore, the modulus of simple rigidity.

The laws of torsion for a uniform cylindrical rod are expressed concisely by the relation, $T = \frac{\pi u r^4}{2l}$, given in Art. 98. They may,

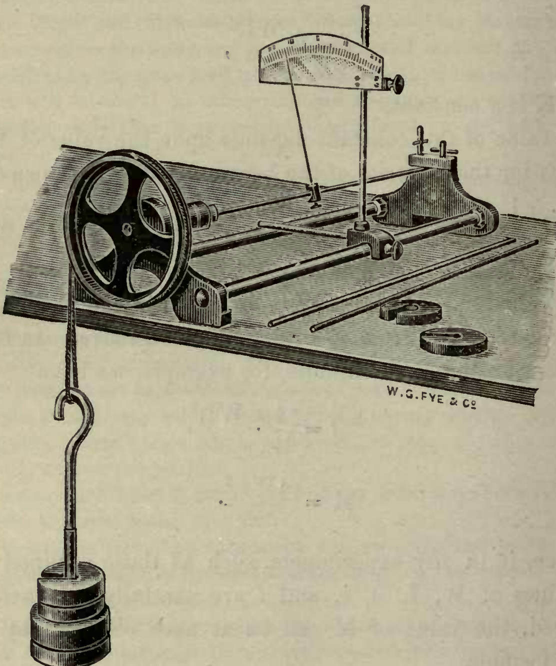

Fig. 122.

however, be established experimentally in a simple manner by means of the apparatus shown in Fig. 122. A long cylindrical rod is mounted in a horizontal position, so that one end is rigidly fixed in a socket, while the other end is free for twisting, and carries a wheel or pulley by means of which a twisting couple can be applied. This couple is applied by hanging a weight from the rim of the wheel. The cord carrying the weight is attached at one end to a point on the rim, and is then coiled once or twice round the grooved rim, so that the free end to which the weight is attached hangs vertically downwards at one side of the wheel. The weight thus acts vertically downwards, in all the positions of the wheel, at the extremity of a horizontal radius, and the moment of the twisting couple thus applied to the rod is, therefore, measured by the product of the weight into the radius of the wheel.

The angle of twist produced in any length of the rod by a given twisting moment is measured by means of the movable pointer and the circular scale shown in the figure. This pointer and scale can be set up at any distance from the fixed end of the rod, and the twist produced in this length of the rod is indicated directly by the deflection of the pointer on the scale.

It should be noted that the moment of the couple which produces torsion in a rod, or the moment of the couple which a rod is able to exert in virtue of torsion imposed on it, is usually called the **torque** applied to, or exerted by, the rod.

The following experiments indicate, in a general way, how this piece of apparatus may be used to determine the experimental laws of torsion for a cylindrical rod.

The angle of twist produced in a circular rod of given material must evidently depend upon (i) the torque applied, (ii) the length of the rod, and (iii) the radius of the rod. It is necessary, therefore, to find by experiment how the twist depends upon each of these three factors separately when the other two are constant.

Experiment 21.—Find how the twist produced in a given rod varies with the torque applied to the rod.

Set up the pointer and scale at a point on the rod near the free end, and measure carefully the angle of twist produced by a number of different torques. The torque should be varied by gradually increasing the weight carried by the torque wheel, and care should be taken not to exceed the limit of perfect recovery for the rod.

It will be found, on tabulating and reducing results, that **the twist produced is directly proportional to the torque applied.**

Experiment 22.—Find how the twist produced depends upon the length of the rod.

Apply a convenient fixed torque to the rod. Then measure the twist produced by this torque in different lengths of the rod by setting up the pointer and scale at different distances from the fixed end of the rod.

It will be found, on tabulating and reducing the results, that **the twist produced by a given torque is directly proportional to the length twisted.**

It should be noted that this means that **the twist per unit length** in a twisted rod is **constant** for a given torque.

Experiment 23.—Find how the twist produced depends upon the radius (or diameter) of the rod.

For this experiment it is necessary to obtain a number of rods of exactly the same material but of different diameters, or to turn the same rod down so as to obtain from it in succession rods of smaller and smaller diameter.

The diameters of the rods should vary within comparatively narrow limits—an extreme ratio of 4 : 5 or 5 : 6 is small enough—and should be measured with the greatest possible accuracy by means of a good screw gauge. Measure the twist produced for the same length by the same torque for each rod in turn.

It will be found, on tabulating and reducing the results, that **the twist produced is inversely proportional to the fourth power of the radius of the rod.**

It follows from the results of these experiments that the twist produced in a cylindrical rod of given material is (i) directly proportional to the torque applied, (ii) directly proportional to the length of the rod, and (iii) inversely proportional to the fourth power of the radius (or diameter) of the rod.

These may be taken as **the experimental laws of torsion** for a cylindrical rod of given uniform material.

Hence, if Q denote the torque applied to a cylindrical rod of length l and radius r, and θ denote the twist produced in the rod, we may write

$$\theta \propto \frac{Ql}{r^4},$$

or

$$\theta = K\frac{Ql}{r^4},$$

where K is a constant.

The value of this constant depends upon the material of the rod, and also upon the fact that the rod is of circular cross section.

If this relation is written in the form

$$Q = \frac{r^4}{Kl}.\theta,$$

it will be seen that the torque, which will produce unit angular twist, is given by

$$\frac{Q}{\theta} = \frac{r^4}{Kl}.$$

But $\frac{Q}{\theta}$ is the quantity denoted by T in the relation,

$$T = \frac{\pi n r^4}{2L},$$

as explained in Art. 98. It follows, therefore, that

$$\frac{r^4}{Kl} = \frac{\pi n r^4}{2l},$$

or

$$K = \frac{2}{\pi n}.$$

Experiment 24.—Set up the apparatus shown in Fig. 122, and measure for a given rod the value of θ for a measured torque Q. Then measure l and r for the rod, and calculate from the relation, $\theta = \frac{KQl}{r^4}$, the value of K for the rod. Then from the relation, $K = \frac{2}{\pi n}$, find the value of n, the modulus of simple rigidity for the material of the rod.

The laws of torsion may be established experimentally by experiments with wires if the apparatus shown in Fig. 123 is used. The method of applying torque to the wire is plainly indicated in the figure.

It is frequently necessary in engineering practice to measure the torque transmitted by a cylindrical shaft. It will be seen, from what has been said above, that this can be done for a shaft of given material and diameter, by simply measuring the twist on a measured length of the shaft. Thus, if θ is found to be the twist on a length l of the shaft, we have

$$Q = \frac{\pi n r^4 \theta}{2l},$$

where Q denotes the torque on the shaft, r the radius of the shaft, and n the modulus of simple rigidity of the material of the shaft.*

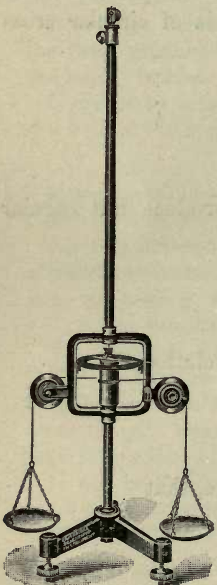

Fig. 123.

102. **Poisson's Ratio.**—When a solid is stretched in any direction it undergoes elongation in that direction and contraction in every direction at right angles to it within the limits of elasticity. The elongation per unit length may be taken as a measure of the elongation strain, and, in the same way, the contraction per unit length may be taken as a measure of the contraction strain. The magnitude of each of these strains is proportional to the stretching stress, but their ratio is constant whatever the stress, and the ratio of the contraction strain to the elongation strain measured in this way is known as **Poisson's ratio.** Thus, if a cylindrical rod of length L and diameter d undergo, when stretched by any load within the

* It must be remembered that θ in this formula is in circular measure.

limits of elasticity, an elongation l in the direction of its length, and a contraction δ laterally at right angles to its length, the *elongation per unit length* is $\frac{l}{L}$, the contraction per unit length is $\frac{\delta}{d}$, and Poisson's ratio is given by the ratio $\frac{\frac{\delta}{d}}{\frac{l}{L}}$.

Hence, if the elongation per unit length, which accompanies stretching, be denoted by α, and the corresponding lateral contraction per unit length by β, and if Poisson's ratio be denoted, in the usual way, by σ, we have

$$\sigma = \frac{\beta}{\alpha}.$$

Imagine a cube of *unit edge* and *unit volume* to be stretched in a direction parallel to one of its edges, and let the elongation per unit length be denoted by α, and the corresponding lateral contraction by β. The dimensions of the stretched cube will now be $(1 + \alpha)$, $(1 - \beta)$, $(1 - \beta)$, and its volume will, therefore, be given by $(1 + \alpha)(1 - \beta)(1 - \beta)$, or $(1 + \alpha - 2\beta)$, if the quantities α and β are assumed to be very small.

Now the cube is of *unit volume* before stretching, so that if no change of volume is produced by stretching, we must have

$$(1 + \alpha - 2\beta) = 1,$$

or

$$\alpha = 2\beta,$$

or

$$\frac{\beta}{\alpha} = \frac{1}{2}.$$

That is, if stretching were simply a change of shape unaccompanied by change of volume, the value of Poisson's ratio for any material should be $\frac{1}{2}$, as found above.

Experiment shows, however, that the value of Poisson's ratio is always less than $\frac{1}{2}$, and differs for different materials. This proves that stretching generally produces an increase of volume, and is, therefore, a compound strain involving change of volume as well as change of shape.

A method of determining Poisson's ratio experimentally is indicated in the experiment given below.

Experiment 25.—Find the value of Poisson's ratio for indiarubber. Get a cord of good rubber about half a metre long and a centimetre in diameter, and set it up so that it can be stretched to nearly double its length.

When the cord is set up put on a sufficient straightening load, mark off by very fine transverse lines a length of about 30 cms. in the middle of the cord. Measure this length exactly, and measure also very carefully the average diameter of the cord over this length. Measurements of the diameter should be made with a good screw gauge every centimetre along the length, but care must be taken not to compress the rubber in making the measurements. When these measurements are taken, load the cord so as to stretch it to nearly double its initial length; then measure again the length between the two transverse lines on the cord and the mean diameter of this part of the cord.

From these measurements find the elongation per unit length and the corresponding lateral contraction per unit length, and calculate Poisson's ratio from the results obtained.

The value of Poisson's ratio for indiarubber is about 0·45.

The value of Poisson's ratio for any material can be derived theoretically from the value of Young's modulus and the modulus of simple rigidity for that material, so that in cases where it cannot be determined experimentally its theoretical value can be calculated from the values of these two moduli.

The value of Poisson's ratio for a few materials is given below.

103. **Table of Elastic Constants.**—The more important elastic constants for a few typical materials are given below in C.G.S. units.

It must be understood that the constants for any given material vary within somewhat wide limits for different specimens. It is found that the value of each constant depends to a considerable extent on the mechanical and thermal treatment to which the material has been subjected.

Material.	Young's Modulus. M. [10^{11} dynes per sq. cm.].	Simple Rigidity. n. [10^{11} dynes per sq. cm.].	Volume Elasticity. k. [10^{11} dynes per sq. cm.].	Poisson's Ratio. σ.
Steel,	20 to 22	8 to 9	13	·27
Iron (wrought),	19 to 21	7·5 to 8·5	13	·30
Iron (cast),	7 to 11	2·5 to 4	9	·35
Copper,	8 to 11	3 to 4	12	·40
Brass,	8 to 10	3 to 3·5	13	·35

It is interesting to note that the value of M for any material should, according to theory, be greater than twice, and less than three times the value of n for the same material. The values of k and σ given above are only approximate mean values.

104. **Hooke's Law.**—It will be seen that in every form of strain considered in the foregoing articles the strain under given conditions is directly proportional to the applied stress, provided the material is not strained beyond the limits of elasticity.

Thus, in volume compression, the volume strain is proportional to the applied stress; in shearing, the shearing strain is directly proportional to the shearing stress; in stretching, the elongation is directly proportional to the stretching force; in bending, the bending deflection is directly proportional to the applied load; and in torsion, the angle of twist is directly proportional to the applied torque.

This general law of the relation between stress and strain was first established experimentally by Hooke, and is known as **Hooke's law.**

Hooke, in order to ensure priority in the discovery of the law, first stated it in an anagram. When he had fully established the law by experiment he gave *ut tensio sic vis* as the solution of the anagram. This phrase concisely expresses Hooke's law.

105. **Elastic After Effects.**—It has already been stated that when a body is strained in any way beyond its elastic limit the strain produced by any given stress gradually increases with the time for which the stress is applied, and that this effect becomes more and more marked the further the limiting stress is exceeded.

It is found, however, that the same effect can be observed for strains within the elastic limits. If any given stress less than the limiting stress is applied for a very long time, careful observation shows that the strain gradually increases with the time. This is one of the time effects connected with stress and strain which are included under the general term **elastic fatigue.**

Another similar effect is the temporary or permanent "set" which a strained body may retain after being relieved from the stress to which the strain is due. It has been explained that when a body is strained in any way beyond its elastic limit it does not recover its original size or shape completely, but receives a *permanent set* which increases as the excess of the applied stress over the limiting stress increases, and also as the time for which a given stress is applied.

It is found also that when a body is strained well within its elastic limits it does not completely recover its original size or shape the instant the stress is removed, but receives a small *temporary set* from which it recovers more or less completely with time.* The amount of this temporary set increases with the stress, and is found, for a given stress, to increase with the time for which the stress acts.

It is found, too, that the time for which the strain lasts, affects not only the magnitude of the temporary set, but also, in a marked degree, the time which this set takes to disappear. For example, if a body is subjected to a strain of any kind

* It is probable that there is in *all* cases of strain a small residual permanent set which increases with the stress and the duration of the strain.

for a long time, and then, in immediate sequence, to a reversal of this strain for a short time, the recovery from the short strain may be complete before the recovery from the long strain, and the recovery effects may thus *appear* to take place in the reverse order of the strains.

From what has been said above, it will be seen that the elastic limits for any material are merely the limits within which the permanent set after strain is negligibly small.

It is probable that in every case of strain some degree of set attends recovery from the strain, and that the *initial set*, at the instant the stress is removed, decreases with time to a small permanent *final set*. When the stress and strain are small the final set is negligibly small, but at a certain arbitrary point it becomes appreciable, and the elastic limit is said to be reached. The limiting stress at which it is reached in any case thus depends upon the limit at which the permanent set ceases to be negligible under the prevailing conditions. It thus appears that no material may be considered to be perfectly elastic except for infinitely small strains.

CHAPTER XVIII.

HYDROSTATICS.

106. **Pressure in a Liquid Neglecting the Weight of the Liquid.**—It has already been explained that if a mass of liquid, whose weight is neglected, is in equilibrim throughout its mass, it must be subject to a normal pressure uniformly distributed over its surface boundary. This statement may be established experimentally, for if we imagine a mass of liquid to be enclosed in a vessel having its walls fitted at a number of points with pistons working in liquid-tight, frictionless collars, as indicated diagrammatically in Fig. 124, it will be found that equilibrium is possible only when the pressure per unit area impressed on the liquid by means of the pistons is exactly the same for each piston. If, when equilibrium exists, the pressure on the unit area on any one piston is increased by any given amount, the pressure per unit area *on every other piston* must be increased by an exactly equal amount.

It has been explained too in the same article that under the conditions considered the normal pressure per unit area at the boundary surface of the liquid is the **pressure in the liquid**, and that this pressure is the same at all points and in all directions at any given point. This result may be established as follows:—Let ABCDE in Fig. 125 represent a mass of liquid in equilibrium under a uniform normal pressure over its boundary surface. Then, since the whole mass is in equilibrium, *any portion* of it, such as BCF, is in equilibrium, and the normal pressure per unit area over the boundary surface CFDC must, therefore, be uniform. That is, the normal pressure per unit area at any point F, in the boundary surface CFD, is the same as the normal pressure at any point in the outer boundary surface of which CD is a part. Now, the surface CFD, separating the portion CFD from the remainder of the mass of liquid, *may be taken through any point in the liquid*, and the direction of the normal to the surface at that point may have any direction we please in the liquid; it follows that the pressure in the liquid is equal to the normal boundary pressure per unit area, and is the same at all points in the liquid, and in all directions at any given point.

It follows from this that any pressure that may be impressed

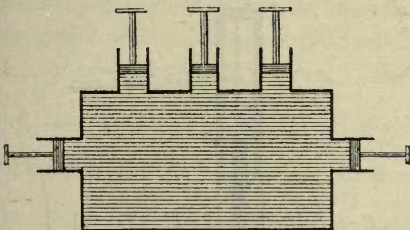

Fig. 124.

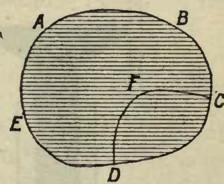

Fig. 125.

upon a liquid, is transmitted by the liquid unchanged in value to every portion of the surface with which the liquid is in contact. This principle was first stated by Pascal, and is sometimes known as *Pascal's law*.

It will be clear from what has been said that if p denote the pressure in a liquid, the total normal pressure exerted by the liquid on any surface of area A with which it is in contact, must be given by Ap. If p is in dynes per square centimetre, and A in square centimetres, the total normal pressure Ap will be given in dynes.

It should be noticed that normal pressure is not here considered as a vector quantity. The total normal pressure on any surface is, in this case, where the pressure in the liquid is uniform, merely the product of the normal pressure per unit area into the area of the surface; or, if we consider the surface divided up into a very large number of very small elements of area, the total normal pressure on the surface is simply the *arithmetical sum* of the normal pressures on these elements of area, without consideration of the form of the surface, or the direction of the normal at any point on it.

107. **The Hydraulic Press.**—The hydraulic press is an important application of the principles explained in the foregoing Article. It consists essentially, as shown in Fig. 126, of

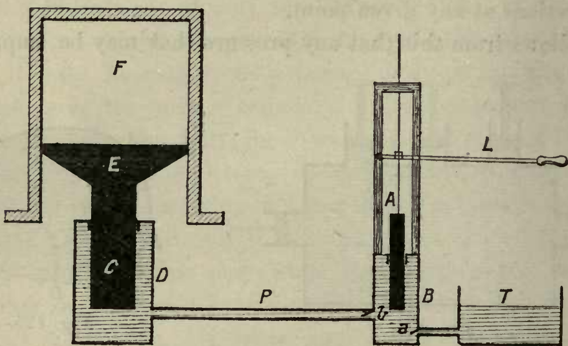

Fig. 126.

two communicating vessels B and D, both of which are fitted with pistons A and C, working in water-tight collars.

The two vessels and the communicating pipe P are made sufficiently strong to stand a very high pressure, and the diameter of the piston or *ram* C in the larger vessel is considerably greater than that in the other vessel.

The vessel B communicates with a tank of water, T, so that when the piston A is worked by means of the lever L, or by other suitable mechanism, water is pumped from the tank T to the vessel B, and forced from the vessel B into the vessel D. Valves at *a* and *b* enable the vessel B and the piston A to act as a force-pump, as explained in Art. 132.

It will be seen that with this arrangement it is possible, by exerting a comparatively small downward force on the plunger A, to cause the head E of the ram C to exert an enormously greater upward force on any substance enclosed in the strongly-constructed chamber F, of which the head forms the floor, and in which it is free to move up and down.

Thus, if the downward force exerted on the plunger A in forcing the water from B into D is denoted by P, and if a denote the area of cross-section of the piston, then $\frac{P}{a}$ denotes the pressure impressed on the water. That is, $\frac{P}{a}$ is the pressure in the water, and is, therefore, the pressure exerted by the water on the surface of the ram C. Hence, if A denote the area of cross-section of the ram C, the upward force exerted on the ram is $A \cdot \frac{P}{a}$, or $\frac{A}{a} \cdot P$. If d denote the diameter of the plunger A, and D the diameter of the ram C, then $\frac{A}{a} = \frac{D^2}{d^2}$, and the force with which the ram is pushed upwards is $P \cdot \frac{D^2}{d^2}$, where P is the downward force exerted on the plunger A. It follows from this result that if D is greater than d, the upward force on the ram may be very much greater than P. For example, if A is 2 inches in diameter, and C 20 inches in diameter, and a force of 100 pound-weights is exerted on A, the upward force exerted on C would evidently be (100×100) pound-weights, or nearly 5 tons weight.

It is evident that the pressure in the water filling the vessels B and D and the communicating pipe P must be very great when the press is in action. It is necessary, therefore, that these vessels and the pipe should be made of sufficient strength to resist extremely high internal pressure. The chamber F, in which substances such as cotton are compressed for packing, must also be very strongly constructed to resist the pressure exerted on the material by the head of the ram as it is forced upwards.

The hydraulic press invented by Pascal and improved by Bramah is sometimes called the **Bramah Press.**

The principle utilised in this press is applied in a great many other ways for industrial purposes. In most cases where very great force has to be exerted at a particular point, *hydraulic power* may conveniently be used. Thus, lifts are worked, dock gates are opened and closed, movable bridges are lifted and lowered, and other similar operations are performed by hydraulic power.

A great advantage of the hydraulic method is that the power can be transmitted from a central station to the points at which work has to be done by means of strong communicating pipes similar to the pipe P in the hydraulic press.

108. **Equilibrium of a Liquid under the Action of Gravity.**—When a liquid is at rest under ordinary conditions in a vessel of any kind the surface of the liquid must evidently be *horizontal.* For, if the surface is not horizontal as at the point A, in Fig. 127, the weight of a molecule of the liquid at A may be resolved into two rectangular components along the normal to the surface, and along the tangent to the surface at that point; the component tangential to the surface will, however, cause the molecule to move in the direction in which it acts, and the liquid cannot, therefore, be in equilibrium. That is, the liquid at the surface cannot be in equilibrium unless the surface is everywhere horizontal or at right angles to the direction in which the weight of every molecule acts.

This result is true, however much the free surface is subdivided by the form of the containing vessel. For example, if a vessel, such as that shown in Fig. 126, consisting of a number of parts, *which are in free communication with each other,* is filled with water, the free surface exposed to the air will be at the same level in each part, and if a hole were made at D and a tube inserted there the water would rise to the same level in this tube as in the other portions of the vessel. This is what is meant by saying that water—or any other liquid—always "finds its own level."

It will be seen, too, that if the liquid is in equilibrium throughout its mass the pressure in the liquid must be the same at every point in the same horizontal plane. For, if the pressure at any point A in the horizontal plane HH, in Fig. 128, is greater than the pressure at any other point B in the same plane, then, *since* A *and* B *are at the same level,* liquid will flow from A to B, and the liquid cannot be in equilibrium throughout its mass. That is, if the mass of liquid is in equilibrium throughout, the pressure in the liquid is the same at

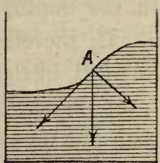

Fig. 127.

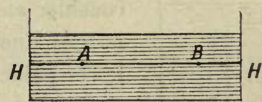

Fig. 128.

every point in the same horizontal plane. Since the surface of the liquid is also a horizontal plane this result evidently means that *the pressure is the same for every point at the same depth in the liquid.*

The pressure due to the weight of the liquid at any point at a given depth in the liquid is readily found. The pressure at any point in the liquid is the same in all directions at that point, and is the same for all points at the same depth. If, therefore, we find the pressure on a unit of area taken *horizontally* at the given depth we get the pressure in the liquid at any point at this depth. Now the pressure *due to the weight of the liquid* on a *horizontal* unit of area at any depth in the liquid is evidently the weight of the column or prism of liquid which stands vertically on this unit of area. Hence, if h denote the depth considered, the *volume* of the column standing on this unit of area is $(h \times 1)$ or h units, and if d denote the density of the liquid, the *mass* of the column is denoted by hd, and the *weight* of the column by hdg, where g denotes the acceleration due to gravity.

That is, the pressure at any point at a depth h in a liquid of density d is given by hdg, where g is the acceleration due to gravity. For example, the pressure at a depth of 20 cms. in mercury of density 13·6 grammes per cubic centimetre is $(20 \times 13{\cdot}6 \times 981)$ dynes per sq. cm., or 266,832 dynes per sq. cm.

It is obvious from this result that the pressure at any depth in the liquid increases with the depth, and is proportional to the depth. If, therefore, we consider any two points, A and B, in the same *vertical* line, as in Fig. 129, the pressure at the lower point B is greater than the pressure at the upper point A, but this difference of pressure does not disturb the equilibrium of the liquid by causing a flow from B to A, for it is exactly balanced by the weight of the liquid between A and B.

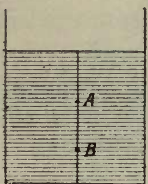

Fig. 129.

If the surface of a liquid at rest under the action of gravity is exposed to any uniform external pressure (such as the atmospheric pressure) this pressure is by Pascal's law exerted at all points in the liquid, and the pressure at any point in the liquid will, therefore, be the sum of this pressure and the pressure due to the weight of the liquid at that point. Thus, if P denote the external pressure per unit area on the surface, the pressure at any point at a depth h in the liquid is given by $(P + hdg)$.

The total normal pressure exerted by a liquid at rest under the action of gravity on any aurface in contact with it cannot evidently be obtained, as in Art. 106, by multiplying the area of the surface by the pressure in the liquid, for this pressure is not constant, but varies, as we have seen, with depth below the surface of the liquid. Hence, in order to find the total normal pressure on any surface in contact with the liquid, it is necessary to divide the area of the surface into an infinite number of infinitely narrow *horizontal* strips, and after finding the normal pressure on each strip to take the sum of these pressures as the total normal pressure on the surface.

Thus, if the areas of the horizontal strips into which the surface is divided, are denoted by $a_1, a_2, a_3 \ . \ . \ . \ .$, and the pressures in the liquid at the depths at which these strips are taken are denoted respectively by $P_1, P_2, P_3 \ . \ . \ . \ .$, then, since the pressure in the liquid is the same for all points in the same horizontal strip, the total normal pressure in the surface must be given by

$$N = a_1p_1 + a_2p_2 + a_3p_3 + \ . \ . \ . \ . \ ,$$

or $$N = \Sigma\,(ap).$$

The general application of this method in any given case involves mathematical difficulties which we cannot here consider. It can be shown, however, that the result obtained above is equivalent to the statement that the total normal pressure on any surface in contact with the liquid is equal to the weight of a column of the liquid, whose area of cross-section is the area of the surface in contact with the liquid, and whose length is equal to the depth of the *centre of gravity of the surface** below the surface of the liquid.

Example.—A trough of triangular cross-section, as shown in Fig. 130, is filled with water. If the length of the trough is 10 feet,

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* That is, the centre of gravity of the surface considered as an infinitely thin uniform lamina.

and the dimensions of its cross-section are as given in the figure, find the total normal pressure on one side and on one end of the trough, the weight of 1 cubic foot of water to be taken as 62·5 pound-weights.

The area of either side of the trough is (10×2) or 20 square feet.

The depth of the centre of gravity of the side is 1 foot.

The total normal pressure on either side is, therefore, the weight of a column of water 20 square feet in cross-section and 1 foot in length, or the weight of 20 cubic feet of water. That is, the total normal pressure on each side is $(20 \times 62{\cdot}5)$ or 1250 pound-weights.

Similarly, the area of either end is (2×1) square feet or 2 square feet, and the depth of the centre of gravity of the triangular surface is $\frac{2}{3}$ foot.

The total normal pressure on each end is, therefore, the weight of $(2 \times \frac{2}{3})$ cubic feet of water. That is, the total normal pressure is $(\frac{4}{3} \times 62{\cdot}5)$ pound-weights, or $83\frac{1}{3}$ pound-weights.

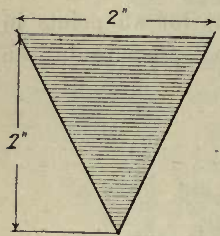

Fig. 130.

It must be remembered that the total normal pressure on any surface is not a single force acting normally to the surface, but a pressure distributed over the whole of the surface in contact with the liquid. In the imaginary case of a liquid without weight this pressure is distributed uniformly over the surface, but in the ordinary case of a liquid under the action of gravity the pressure at any point on the surface increases as the depth of the point below the surface increases, so that the total normal pressure cannot be uniformly distributed over the surface.

If we consider, not the total normal pressure on a surface in contact with a liquid, but the **resultant effect** of the pressure acting on the surface, and imagine this resultant effect to be produced by an equivalent single force acting at a point on the surface, the magnitude of this equivalent single force gives the push or **thrust** exerted on the surface, and the point at which this force would act on the surface is called the **centre of pressure** for the surface.

109. **The Principle of Archimedes.**—Let a mass of liquid be at rest under the action of gravity in any vessel, and consider the equilibrium of any portion of the liquid taken, as at A in Fig. 131, in the interior of the liquid, and supposed to be separated from the rest of the liquid by an **imaginary** boundary surface, indicated by the dotted line on the figure.

This portion of the liquid is in equilibrium, and the only forces acting on it are the pressure exerted on it all over its boundary surface by the surrounding liquid, and its weight. The resultant effect of the normal pressure distributed over the boundary surface must, therefore, be equal and opposite to the weight of the liquid enclosed by the surface. The weight of

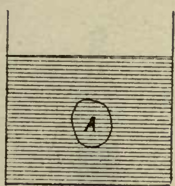

Fig. 131.

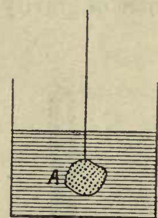

Fig. 132.

the liquid enclosed by the surface acts vertically downwards through the centre of gravity of this portion of the liquid; the resultant effect of the normal pressure on the boundary surface must, therefore, be equal to the weight of the liquid enclosed by the surface, and must act vertically *upwards* through the centre of gravity of this portion of the liquid. That is, the resultant **thrust** on any portion of the liquid supposed to be separated from the rest by an imaginary boundary surface, is equal to the weight of the liquid enclosed by the surface, and acts vertically upwards through the centre of gravity of this portion of the surface.

Now, if a solid body is immersed in a liquid at rest under the action of gravity, as at A in Fig. 132, the resultant **upthrust,** due to the pressure of the surrounding liquid on the surface of the body, must evidently be exactly the same as if the space occupied by the body were occupied by the liquid. For, it is clear that the resultant thrust on the surface, due to the normal pressure exerted on it by the surrounding liquid, depends only on the extent, form, and position of the surface, and on the density of the surrounding liquid. It evidently cannot be affected in any way by the nature of the material *within* the surface.

The resultant upthrust of the liquid on the solid body immersed in it is, therefore, equal to the weight of the liquid displaced by the body, and acts vertically upwards through the centre of gravity of the displaced liquid.*

Fig. 133.

The principle here stated is **the principle of Archimedes.**

It follows from this principle that if the weight of a body immersed in a liquid is greater than the weight of the liquid it displaces, the body sinks, but if the weight of the body is less than the weight of the liquid it displaces, the body rises to the surface and floats.

It follows also that if a body is weighed first in air in the usual way, and then weighed again as it hangs immersed in liquid in which it sinks, the weight observed in the second case must be less than the weight in the first case, for in the

* That is, the centre of gravity of the liquid that would fill the space actually occupied by the body. It is usual to call this the centre of gravity of the displaced liquid.

second case the body is partially supported, or buoyed up, by the upthrust due to the pressure of the liquid in which it is immersed. The *apparent weight* of a body immersed in any liquid in which it sinks will, therefore, be less than its *true weight in vacuo** by the weight of the liquid which it displaces.

This result is a specially important one, and has a number of important applications.

Experiment 26.—Show that the apparent weight of a body in

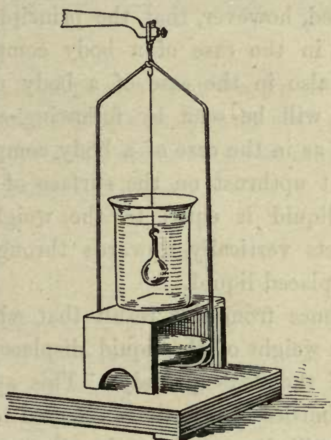

Fig. 134.

water is less than its true weight by the weight of water which it displaces.

Get a glass sinker similar to one of these shown in Fig. 133, and made of solid glass rod, or of a tube or bulb weighted with mercury as shown in the figure. Weigh the sinker carefully in the usual way and record the weight. Then hang it from the suspension hook of the scale pan by a single silk fibre or a very fine wire, and weigh it as it hangs fully immersed in water. The arrangement of the balance for making this weighing is shown in Fig. 134. The beaker containing the water rests on the small stool or bridge which spans the scale pan, so that the body can

* See Art. 125.

hang immersed in the liquid without interfering with the free movement of the pan as the beam of the balance oscillates.

Record the apparent weight thus observed. The difference between this weight and the weight first observed should be very nearly* equal to the weight of water displaced by the sinker. Now take the sinker and find its volume as accurately as you can by the method of displacement explained in Art. 16. Then calculate (1 c.c. of water weighs approximately 1 gramme) and find by actual weighing the weight of this volume of water. It will be found to agree very closely with the *apparent loss of weight* of the sinker in water.

It must be noted, however, that the principle of Archimedes applies not only in the case of a body completely immersed in a liquid, but also in the case of a body *partially immersed* in a liquid. It will be seen by following exactly the same line of argument, as in the case of a body completely immersed, that the resultant upthrust on the surface of a body partially immersed in a liquid is equal to the weight of the liquid displaced, and acts vertically upwards through the centre of gravity of the displaced liquid.

It follows at once from this result that when a body *floats* in any liquid the weight of the liquid displaced must be equal to the weight of the floating body. This application of the principle of Archimedes is sometimes referred to as the principle of *flotation*. It will be seen that in all cases where a body floats on a liquid, the *displacement* of liquid by the floating body depends on the weight of the body, for the weight of the displaced liquid must in every case be equal to the weight of the body.

Experiment 27.—Take a test-tube and pour a sufficient quantity of mercury into it to make it float in water in a vertical position immersed to rather less than half its length.

Float the test-tube in water in a graduated measuring vessel so that any *change* in the volume of water displaced by the tube can be measured by means of the graduations on the measuring vessel.

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* See Art. 125.

Note the reading of the water level in the vessel, then pour 20 c.c. of water into the *test-tube* (from another measure), and again note the reading. The difference of these two readings gives the increase in the volume of the displaced water, and it will be found in this case to be 20 c.c., the same as the volume poured into the test-tube. That is, any increase (or decrease) in the weight of the floating body produces an exactly equal increase (or decrease) in the weight of the displaced liquid. This result is in accordance with the principle of flotation.

[It may be noted that a test-tube floating in this way in a measuring vessel may be used as a rough balance. For if any body is placed in the tube and the increased displacement is found to n c.c., it is obvious that the weight of the body is roughly n grammes.]

Horizontal liquid surfaces at rest on the earth conform to the generic curved outline of the globe, and are only approximately horizontal over very small areas. The surface of the ocean or a large lake is distinctly curvilinear.

CHAPTER XIX.

EXPERIMENTAL DETERMINATION OF SPECIFIC GRAVITY AND DENSITY.

110. **Methods Based on the Principle of Archimedes.**—The most convenient and accurate methods of determining the specific gravity of a solid or a liquid are direct applications of the principle of Archimedes.

The specific gravity of a substance has been defined as the ratio of the weight of any volume of the substance to the weight of the same volume of water at 4° C. It is necessary, therefore, in order to determine the specific gravity of a substance to find the weight of a suitable portion of the substance, and then the weight of an exactly equal volume of water at 4° C.

Now, when a solid body is weighed in water its apparent loss of weight, as explained above, gives the weight of the displaced water, and the volume of this displaced water must be exactly equal to the volume of the body which displaces it. Hence, if a **solid body** is weighed in air in the usual way, and then in water, the weight in air gives practically* the true weight of the body, and the difference of the two weights, as the apparent loss of weight in water, gives the weight of an exactly equal volume of water. The specific gravity of the material of the body is given, therefore, at once, by the ratio of the weight of the body in air to its apparent loss of weight in water. That is, if W denote the weight of the body in air, and W′ its weight

* See Art. 125.

in water, then the apparent loss of weight of the body in water is W − W′ and s; the specific gravity of the material of the body is given by

$$s = \frac{W}{W - W'}.$$

It must be noted, however, that this gives the specific gravity of the substance at the temperature of the water in which it is weighed, relative to water at this temperature. If the temperature of the water is 4° C., then s is the true specific gravity of the substance at 4° C.; if, however, the water is at $t°$ C., and the density of water at $t°$ C. relative to water at 4° C. is denoted by σ, then the true specific gravity of the substance at $t°$ C. is $s\sigma$, for if the substance is s times as dense as water at $t°$ C., and water at $t°$ C. is σ times as dense as water at 4° C., it follows that the substance is $s\sigma$ times as dense as water at 4° C.

The water in which the body is weighed should be pure distilled water free from air.

The method of making a determination of specific gravity by this method is indicated in Fig. 134. The body is first weighed in air, in the usual way, and then weighed in water by the arrangement shown in the figure.

The method, in this direct form, obviously applies only to solids which are insoluble in water. If the solid is denser than water there is no difficulty in making the necessary weighings, but if the solid is less dense than water, a *sinker* must be attached to it to make it sink. This complicates the method slightly, but the effect of the sinker is easily eliminated. In order to effect this elimination it is necessary to make three weighings:—(1) To weigh the solid in air in the usual way; (2) to weigh the sinker by itself in water; and (3) to weigh the "combination" of solid and sinker fastened together* in water.

* A piece of platinum, silver, or lead wire, which can be coiled round the solid, forms a convenient sinker.

It will readily be seen that if W_1 denote the weight of the solid in air, W_2 the weight of the sinker in water, and W_3 the weight of the combination in water, then $[(W_1 + W_2) - W_3]$ must be the apparent loss of weight of *the solid* in water, and that s, the specific gravity of the solid, is, therefore, given by

$$s = \frac{W_1}{W_1 + W_2 - W_3}.$$

This gives, as before, the specific gravity of the solid relative to water at the temperature of weighing, and is subject, theoretically, to the correction indicated above, in order to get the specific gravity relative to the water at 4° C.

If the solid is soluble in water then the same general method may be adopted, but, instead of weighing it in water, it should be weighed in some liquid in which it is insoluble, or in a *concentrated* solution of the solid in water. Then the value of s, given by the ratio of the weight of the substance in air to the apparent loss of weight *in the liquid*, is the density of the solid relative to the liquid, and if σ denote the density of this liquid relative to water at 4° C., then $s\sigma$ gives, as explained above, the true specific gravity of the solid.

This method may also be adapted for the determination of the specific gravity of a **liquid.**

The apparent loss of weight of a sinker in any liquid gives exactly the weight of the displaced liquid. Hence, if the *same* sinker is weighed in any given liquid and in water, the apparent loss of weight in the liquid gives the weight of a certain volume of the liquid, and the apparent loss of weight in water gives the weight of an *exactly equal volume* of water. The specific gravity of the liquid will, therefore, be given by the ratio of the apparent loss of weight of the sinker in the liquid to its apparent loss of weight in water. Thus, if W_1 denote the weight of a sinker in air, W_2 its apparent weight in water at 4° C., and W_3 its apparent weight in any liquid, the specific gravity of the liquid is evidently given by

$$s = \frac{W_1 - W_3}{W_1 - W_2}.$$

This method can be adopted with most liquids, for a glass sinker, similar to those shown in Fig. 133, suspended by a fine platinum wire would not be attacked by many liquids.

It must be noted that all the methods referred to above are capable of the highest accuracy. The only measuring operation which they involve is that of weighing, and this is an operation which can be performed with a good balance to a very high degree of accuracy.

111. **The Specific Gravity Bottle.**—The specific gravity of a liquid may be determined conveniently by means of a specific

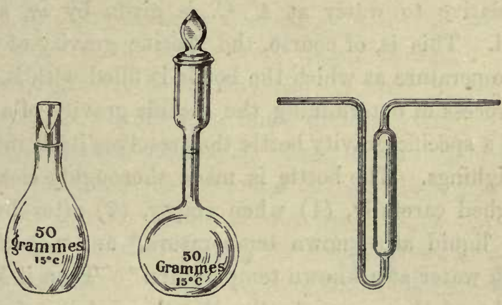


Fig. 135.—Specific Gravity Bottles.

gravity bottle or flask. A common form of this bottle is shown on the left in Fig. 135. It consists of a small flask fitted with a glass-stopper made from a short length of thick-walled tubing of very fine bore. The bottle thus becomes, when the stopper is inserted, a small flask with a very narrow neck formed by the bore of the stopper, and if it is filled with liquid, by first nearly filling it to the brim, and then inserting the stopper so as to cause the liquid to overflow through the bore, there is every certainty that the volume of liquid contained by the bottle at any given temperature will be exactly the same every time it is properly filled.

If, therefore, a specific gravity bottle is filled, first with the liquid, whose specific gravity is to be found, and then with water, and the weight of the liquid which fills the bottle in each case is carefully determined by making the necessary weighings, the ratio of the two weights thus obtained evidently gives the specific gravity of the liquid. The ratio is the weight of a certain volume of the given liquid to the weight of *the same volume* of water, and is, therefore, the density of the liquid relative to water at the temperature of filling the bottle. Hence, if s denote the value of this ratio, and σ denote, as above, the specific gravity of water at the temperature of the experiment relative to water at 4° C., the true specific gravity of the liquid relative to water at 4° C. is given by $s\sigma$, as already explained. This is, of course, the specific gravity of the liquid at the temperature at which the bottle is filled with it.

The process of determining the specific gravity of a liquid by means of a specific gravity bottle thus resolves itself into making three weighings. The bottle is made thoroughly clean and *dry* and weighed carefully, (1) when empty, (2) after being filled with the liquid at a known temperature,* and (3) after being filled with water at a known temperature.* Then if W_1 denote the weight of the empty bottle, W_2 the weight of the bottle after being filled with the liquid at $t°$ C., and W_3 the weight of the bottle after being filled with water at $t°$, the specific gravity of the liquid at $t°$ C. relative to water at $t°$ C. is given by

$$s = \frac{W_2 - W_1}{W_3 - W_1}.$$

* The bottle is most conveniently filled with a liquid at a definite temperature by immersing it in the liquid in a beaker containing the liquid and a thermometer indicating the temperature. The temperature at which the bottle is *filled* in each case should be a little higher than the temperature at which the weighings are made; this prevents loss of liquid by overflow after filling.

Then, if σ denote the specific gravity of water at t° C. relative to water at 4° C., the true specific gravity of the liquid at t° C. is $s\sigma$, as stated above.

It is evident that the weight W_1, and the weight W_3 *for water at 4° C.* may be determined once for all, so that if these weights are known accurately for any particular bottle, it is only necessary in finding the specific gravity of a liquid with that bottle to find W_2 as accurately as possible. The value of $(W_2 - W_1)$ can then be found from the known value of W_1, and $(W_3 - W_1)$ is a known constant for the bottle, so that the true specific gravity relative to water at 4° C. is given at once by the ratio denoted by s above.

In some cases a counterpoise which accurately balances the weight of the clean dry bottle is provided with the bottle. The use of this counterpoise does away with the necessity of *drying* and weighing the bottle every time it is used.

The specific gravity bottle may also be used for finding the specific gravity of a solid in a finely divided state. Thus, the specific gravity of shot, sand, metal filings, insoluble powders, and other similar substances may be found by the method given below.

The specific gravity bottle is not used as in the case of a liquid but as a means of finding the weight of water displaced by the solid substance. The weight of water which completely fills the bottle can be found as explained above. Similarly, the weight of water which fills the bottle *when the solid substance is inside the bottle* can also be found. The difference of these two weights evidently gives the weight of water displaced by the solid substance.

Hence, if W denote the weight taken of the solid substance, W_1 the weight of the clean dry bottle, W_2 the weight of the bottle filled with water, and W_3 the weight of the bottle with the solid substance inside it, and then filled up with water, then $(W_2 - W_1)$ is the weight of water which completely fills the bottle, $[W_3 - (W_1 + W)]$ is the weight of water which fills the bottle when the solid is in the bottle, and the difference of these two weights is, evidently, the weight of water displaced by the solid in the bottle. If this weight be denoted by W′ the specific gravity of the substance is given by

$$s = \frac{W}{W'},$$

for W′ is the weight of a volume of water exactly equal to the volume of the solid substance of weight W.

This result must, if necessary, be corrected for the temperature of the water as explained above.

The process of making a determination by this method is fairly obvious. A convenient quantity of the solid substance is taken—about sufficient to fill the bottle half full—and its weight, W, is determined in the usual way. The weights W_1 and W_2 are then determined as explained above. The bottle is then half filled with water and the solid substance is run into it slowly in such a way that every particle of the substance is wetted by the water, and no air bubbles are entangled in the mass. The substance is then allowed to settle, and when this is over the bottle is, if necessary, filled up with water and the stopper inserted. The weight W_3 can then be determined, and the specific gravity of the substance calculated from the data obtained.

Example.—Find the specific gravity of sand from the following data :—

Weight of sand taken	= 96·4 grammes.
Weight of bottle	= 23·1 ,,
Weight of bottle full of water	= 123·3 ,,
Weight of bottle containing the sand and filled up with water	= 170·3 ,,

From these data we get—

Weight of water which fills the bottle = (123·3 − 23·1) grammes	= 100·2 grammes.
Weight of water in the bottle when the sand is placed in the bottle = 170·3 − (96·4 + 23·1) grammes	= 50·8 ,,

Hence, weight of water displaced by the sand is (100·2 − 50·8) grammes or 49·4 grammes.

The weight of the sand is 96·4 grammes.

The specific gravity of the sand is, therefore, $\frac{96{\cdot}4}{50{\cdot}8}$ or 1·7.

The correction for the temperature of the water is not here made. It is at most a very small correction, and need only be made in very accurate determinations.

[No attempt should be made to work an example of this kind by the use of a formula. Every step should be set out from first principles, but the student must know definitely, from the theory of the method, the successive steps to be taken.]

It will be seen that a determination of the specific gravity by means of a specific gravity bottle involves weighing only, and may, therefore, be made with great accuracy.

112. **Hydrometers.**—An hydrometer is an instrument designed to float vertically in a liquid and constructed to indicate the specific gravity of the liquid, either by the depth to which it sinks when floating in the liquid, or by the weight necessary to make it float immersed to a certain fixed depth in the liquid. In the former case the hydrometer is of the type known as *variable immersion* hydrometers, in the latter case it is called a *constant immersion* hydrometer.

A common form of variable immersion hydrometer is shown in Fig. 136. It consists of a glass bulb and stem weighted with mercury so that it floats with the stem vertical. When floating in any liquid the hydrometer is immersed to a depth which depends upon the specific gravity of the liquid, and since the weight of the displaced liquid is, in any liquid, equal to the weight of the instrument, the depth to which it is immersed must increase as the density of the liquid decreases. It is, therefore, possible by graduating the stem to provide a scale which will indicate the specific gravity of the liquid in which the instrument floats within a range which depends upon the construction of the instrument. Thus, if an hydrometer floats immersed up to a mark at the bottom of the stem in water at 4° C. and up to a mark near the top of the stem in a liquid of specific gravity 0·7, it is evidently possible by graduating the stem between these two marks so as to indicate

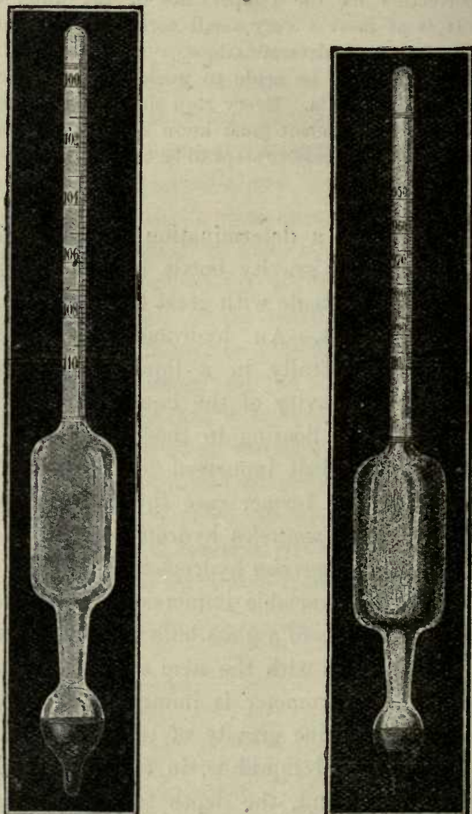


Fig. 136.—Variable Immersion Hydrometer.

any specific gravity between ·7 and 1. Similarly, if an instrument floats immersed up to a mark near the top of the stem in water at 4° C. and up to a mark at the bottom of the stem in a liquid of specific gravity 1·2, it may evidently be graduated to indicate any specific gravity between 1 and 1·2.

In this way by means of a *set* of hydrometers with consecutive or overlapping ranges it is possible to determine the specific gravity of a liquid by simply finding the instrument which floats in it and then reading the graduation which marks the depth to which it floats.

A number of somewhat empirical scales of graduation have been used for various technical purposes; the simplest for general use is that on which the specific gravity of water at 4° C. is marked 1,000, and specific gravities higher and lower than this by correspondingly higher and lower numbers. On this scale the specific gravity corresponding to any division is evidently obtained by dividing the number of the division by 1,000. Thus, if an hydrometer floats in a liquid immersed to a point marked 969 on the scale of the stem, the specific gravity of the liquid is ·969.

Fig. 137. Nicholson's Hydrometer.

The only constant immersion hydrometer in common use is *Nicholson's hydrometer*. A convenient form of this instrument is shown in Fig. 137. It consists of a central hollow cylinder or barrel, C, carrying, in the same axial line, an *upper pan* at A and a *lower pan* at B. The *lower pan* is weighted with lead so that the instrument may float with its axis AB vertical, and a fine line on the straight wire *stem* which carries the upper pan marks the fixed point to which the instrument is immersed in all liquids.

The hydrometer is so constructed that it floats in any liquid with only a portion of the central barrel immersed. It can, however, be sunk in any liquid to the fixed mark on the stem by placing weights in the upper pan, until the mark is seen at the surface of the liquid. This adjustment is best made by first sinking the instrument a little too deeply so that the mark is below the surface of the liquid, and then gradually reducing the weight in the pan until the image of the mark, seen from below by total reflection at the surface, coincides with the mark itself.

When the hydrometer floats in any liquid immersed exactly to the mark on the stem the weight of the instrument together with the weight in the pan is, by the principle of Archimedes, equal to the weight of the displaced liquid. Hence, if the hydrometer floats immersed up to the mark in any given liquid, and then in water at 4° C., it displaces exactly the same volume in each case, and the ratio of the weight of the loaded instrument as it floats in the liquid to its weight as it floats in the water is, therefore, the specific gravity of the liquid. That is, if W denote the weight of the hydrometer, W_1 the weight necessary to sink it to the index mark in the liquid, and W_2 the weight necessary to sink it to the mark in water at 4° C. (or at t° C. if the necessary correction is afterwards made), then, s, the specific gravity of the liquid is given by

$$s = \frac{W_1 + W}{W_2 + W}.$$

The weights W and W_1 may evidently be determined once for all as working constants for the hydrometer for which they are determined.

Nicholson's hydrometer may also be used to determine the specific gravity of a solid substance. The method of using the instrument for this purpose is as follows :—The hydrometer is floated in water in a tall wide jar as shown in Fig. 138. The weight necessary to sink it to the index mark is then found as explained above. The weights are then removed and the piece of solid whose specific gravity is to be found is placed on the pan. The weight of the solid must not, however, be sufficiently great to sink the instrument to the index mark, and weights must, therefore, be added to effect this adjustment. When this adjustment is made and the weight in the pan noted, the solid is removed from the upper pan and placed in the lower pan where it displaces its own volume of water. Since the apparent weight of the solid is less in water than in air the hydrometer will now float less deeply immersed, and it is necessary, in order to sink it again to the index mark, to add to the weight in the upper pan. This is done and the weight in the pan again noted.

The specific gravity of the solid can now be calculated from the data obtained in this way. Let W_1 denote the weight necessary to sink the hydrometer to the index mark in water, W_2 the weight necessary to sink it when the solid is in the upper pan, and W_3 the weight necessary for the same purpose when the solid is in the lower pan. Then, a very little consideration will show that the weight of the solid is $(W_1 - W_2)$, and the weight of water which it displaces when in the lower pan is $(W_3 - W_2)$. The specific gravity of the solid is, therefore, given by

$$s = \frac{W_1 - W_2}{W_3 - W_2}.$$

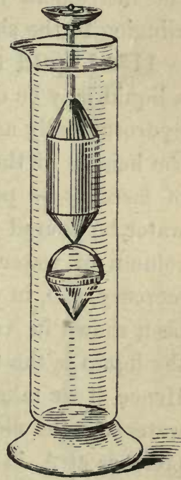


Fig. 138.

The method is not accurate enough to make it necessary to consider the correction for the temperature of the water. It will be seen that the hydrometer serves as a balance for determining the weights to be observed in using the instrument. This is convenient for some purposes, but it will be found in practice that the necessary weights cannot be found with certainty to much less than a decigram.

This hydrometer is mainly of theoretical interest and is used more in the laboratory than anywhere else. It is usually made of brass for use in water for the determination of the specific gravity of solids. It can, however, be made of glass and platinum wire, and may then be used for liquids or solids. In using the instrument a guard disc should be placed over the mouth of the jar containing the liquid to prevent the weights getting into the liquid by falling in accidentally, or by overloading the hydrometer. The disc may be of cardboard, glass, or sheet metal, and must be cut with a narrow radial slot for the stem of the hydrometer. This disc serves also to keep the instrument in the middle of the liquid and prevents it clinging to the sides of the jar.

113. **Liquid Column Methods.**—The specific gravity of a liquid may be determined by balancing a column of the liquid hydrostatically against a column of water, and then comparing the heights of the two balancing columns. Thus, let a quantity of mercury be poured into the bend of a **U**-tube, and then let water be poured on the mercury in one limb of the tube till the column of water, AB, in this limb is balanced by a column of mercury, CD, in the other limb, as shown in Fig. 139. It has been shown in Art. 108 that in a liquid at rest the pressure in the liquid is the same at all points in the same horizontal plane. Hence, if we take a horizontal plane through the junction of the water and the mercury at A, in the limb AB, it follows that the pressure at A, in the limb AB, is the same as the pressure at C, in the limb CD.*

Now, if h_1 denote the height of the column AB, and d_1 the density of the water of the column, the pressure at A due to

* In applying this principle in a case of this kind it is very important to notice that the principle applies only when the liquid *below* the horizontal plane considered is one and the same liquid throughout. For example, if we take any horizontal plane between C and D in the figure, the pressure is *not* the same at points in this plane in both limbs of the tube. The essential point is that the pressure will be the same at two points in the same horizontal plane only if there is free communication between the points by a path below the plane which passes from one point to the other through one and the same liquid.

the weight of the overlying water is h_1d_1g, as explained in Art. 108. Similarly, if h_2 denote the height of the mercury column CD, and d_2 the density of the mercury, the pressure at C due to the weight of the overlying mercury is h_2d_2g.* If, therefore, the pressure at A is equal to the pressure at C, we must have

$$h_1d_1g = h_2d_2g,$$

or

$$h_1d_1 = h_2d_2.$$

That is,

$$\frac{h_1}{h_2} = \frac{d_2}{d_1}.$$

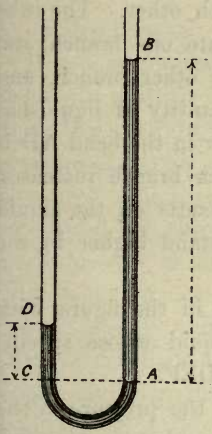


Fig. 139.

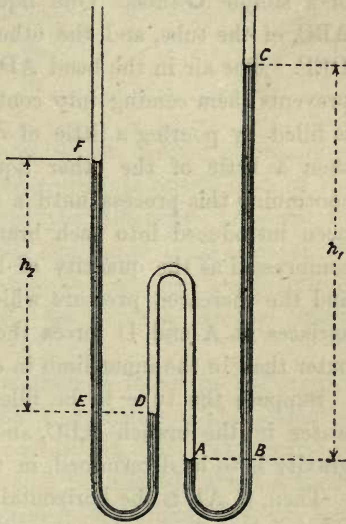


Fig. 140.

But $\frac{d_2}{d_1}$ is the ratio of the density of mercury to the density of water, and is, therefore, the relative density or specific gravity of mercury. That is—

$$\frac{h_1}{h_2} = s,$$

* The external atmospheric pressure impressed on the columns at B and D is the same for both columns, and need not here be considered.

where s denotes the specific gravity of mercury at the temperature of the experiment relative to water also at the temperature of the experiment. If this temperature is observed the usual reduction to 4° C. can be made, but the method is not, under ordinary conditions, sufficiently accurate to make it worth while doing so.

This method may be employed for determining the specific gravity of any liquid, or for comparing the densities of any two liquids, by using a tube of the form shown in Fig. 140 instead of a simple U-tube. One liquid is poured into one branch, ABC, of the tube, and the other liquid into the other branch, DEF. The air in the bend AD separates the two liquids, and prevents them coming into contact with each other. The tube is filled by pouring a little of one liquid into one branch, and then a little of the other liquid into the other branch, and continuing this process until a sufficient quantity of liquid has been introduced into each branch. The air in the bend AD is compressed as the quantity of liquid in each branch increases, and the increased pressure which it thus exerts on the liquid surfaces at A and D forces the liquid to stand higher in the outer than in the inner limb in each branch.

Suppose the tube to be filled, as shown in the figure, with water in the branch ABC, and with a liquid whose specific gravity is to be determined, in the branch DEF.

Then, if AB is the horizontal through A, the pressure in the tube at A is the same as the pressure at B, and the pressure at B is the pressure due to the column of water BC, together with the atmospheric pressure impressed on the surface of the water at C. That is, if h_1 denote the vertical height BC, d_1 the density of the water in the tube, and P the atmospheric pressure impressed on the surface at C, the pressure at B is $(P + h_1 d_1 g)$, as already explained; and this is also the pressure at A when the water is in contact with the air in the bend AD.

In exactly the same way, if DE represents the horizontal through D, it follows that if h_2 denote the vertical height EF, d_2 the density of the liquid in the branch DEF, and P the atmospheric pressure on the liquid surface at F, the pressure at E in the liquid is $(P + h_2d_2g)$, and that this is also the pressure at D where the liquid is in contact with the air in the bend AD. The pressure at A is, however, practically the same* as the pressure at D, for it is the pressure of the air in the bend AD. We may, therefore, write

$$P + h_1d_1g = P + h_2d_2g,$$

or

$$h_1d_1 = h_2d_2.$$

That is,

$$\frac{h_1}{h_2} = \frac{d_2}{d_1} = s,$$

as obtained above.

It may be noticed that the most direct method of obtaining the above result is to note that the pressure due to each of the liquid columns BC and EF gives the difference between the pressure of the air in the bend AD and the external atmospheric pressure. The pressure due to these two columns are, therefore, equal, and we may at once write $h_1d_1g = h_2d_2g$, and so obtain the value of s, as above.

Another application of this method is found in the hydrometer usually known as **Hare's hydrometer.** The essential parts of this hydrometer are shown in Fig. 141. It consists of two straight lengths of glass tubing, BC and EF, about a metre long, set up a short distance apart in a vertical position, and joined at their upper ends by rubber connections, RR, to the T-piece T. The lower ends of the tubes dip into the beakers A and D, and the whole piece of apparatus is supported by a suitable wooden stand. A rubber tube, S, is attached to the stem of the T-piece, and is used to draw air out of the tubes.

* Pressure due to the weight of the air may be neglected except in cases where, as in the atmosphere, very great differences of level have to be considered.

The two liquids whose densities are to be compared are placed in the beakers A and D. Then, on withdrawing air from the tubes, by means of the tube S, the pressure of the air in the tubes decreases; that is, the pressure on the liquid surfaces inside the tubes decreases, and the liquid is forced up into each tube by the excess of the external atmospheric pressure on the

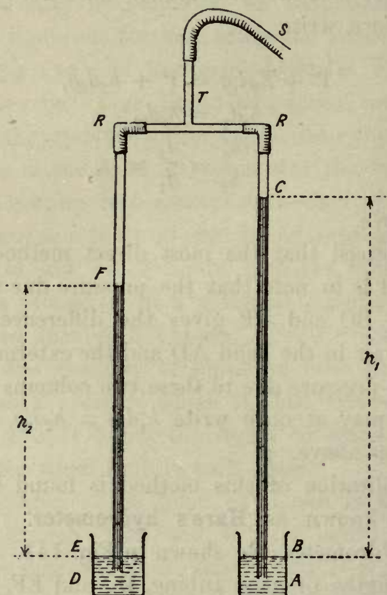


Fig. 141.

liquid surfaces in the beakers outside the tubes, over the pressure *at the same level* inside the tubes. When the liquid rises in each tube to such a height that the pressure in the liquid in the beakers is the same at the same level outside and inside the tube the mass of liquid in each beaker and its tube will again be in equilibrium. Hence, if the tube S is closed with a screw-clip after a quantity of air is withdrawn from the tubes, a column of liquid will be left standing in equilibrium in each tube.

In an experiment with this hydrometer for determining the specific gravity of a liquid, suppose water to be placed in the beaker A and the liquid in the beaker D, and let BC and EF represent the columns of water and liquid respectively, which stand in the tubes after air has been withdrawn by the tube S, and the tube closed.

Then, if h_1 denote the vertical height of the column of water BC, measured from a point, B, inside the tube at the level of the water outside the tube to the upper level at C, d_1 the density of the water, and p the pressure of the air in the bend CRRF, the pressure in the water at the point B is $(p + h_1 d_1 g)$. Similarly, it can be shown that the pressure in the liquid column EF at the point E inside the tube at the level of the liquid outside the tube is $(p + h_2 d_2 g)$, where h_2 denotes the vertical height EF and d_2 the density of the liquid.

Now, the pressure at B and the pressure at E must be equal, for each is equal to the external atmospheric pressure on the surface of the liquid in the beaker. We may, therefore write

$$p + h_1 d_1 g = p + h_2 d_2 g,$$

or

$$h_1 d_1 = h_2 d_2.$$

That is,

$$\frac{h_1}{h_2} = \frac{d_2}{d_1} = s,$$

where s is the specific gravity of the liquid relative to the water in the beaker.

As in the case considered above, this result may be obtained more directly and concisely by stating that the pressure due to each of the columns BC and EF measures the difference between the pressure of the air in the bend CRRF and the external atmospheric pressure. We may, therefore, at once write $h_1 d_1 g = h_2 d_2 g$, or $\frac{h_1}{h_2} = s$, as before.

This method of comparing densities by balancing liquid columns is not in ordinary practice a very accurate method.

It is, however, capable of considerable accuracy when specially accurate methods of measuring the heights of the columns are adopted. A very notable instance of the use of this method is the method of Dulong and Petit for determining the coefficient of absolute expansion of mercury,* by comparing the density of mercury at different temperatures with its density at 0° C. The same method was also employed by Thiesen in one of the best of the more recent determinations of the density of water at different temperatures relative to water at 4° C.

It is important, too, to note that the usual method adopted in physical measurements for measuring pressure or difference of pressure is to balance the pressure to be measured by a liquid column, and then to calculate the magnitude of the pressure from the height of the column and the density of the liquid.

114. **Absolute Density of Water.**—It will be understood from what has been said above that the best method of determining the absolute density of any liquid is to determine its specific gravity relative to water at 4° C. Then, if the absolute density of water at 4° C. is known, the absolute density of the liquid can be calculated. Thus, if s denote the specific gravity of the liquid, and d the absolute density of water at 4° C., sd is the absolute density of the liquid.

The absolute density of water at 4° C. is thus an important physical constant, and a vast amount of careful and laborious research has been expended on the experimental determination of this constant.

The results of several important determinations made in recent years agree in placing the absolute density of pure, air-free water, at the normal atmospheric pressure, and at the temperature of its maximum density (4° C.) between 0·99995 and 0·99996 in grammes per cubic centimetre.

115. **Indirect Methods of Measuring Capacity and Volume.**—The most convenient and most accurate method of

* See *Heat*, Art. 25.

finding the capacity of a vessel is to find, by direct weighing, the weight of water or mercury which fills the vessel at a known temperature. The volume of the mass of liquid can be at once calculated from its mass and density at the observed temperature of filling, and this gives the capacity of the vessel at this temperature. Thus, if m denote the mass of the liquid which fills the vessel, and d its density at the observed temperature, then $\frac{m}{d}$ is its volume, and this is also the internal volume or capacity of the vessel. Water is used when the capacity to be measured is large, and mercury when the capacity is small.

Experiment 28.—Find the capacity per cm. length of the bore of the given tube of fine bore. Clean and dry the tube carefully. Draw a long thread of clean, dry mercury into the tube and measure the length of thread as accurately as possible. Observe the temperature at which this measurement is made. Then run the mercury out of the tube into a weighed porcelain crucible, and find its weight by careful weighing on the balance. From the data thus obtained calculate out the required capacity.

Example.—In an experiment of this kind the following data were obtained. Calculate the capacity per cm. length, and also mean diameter, of the bore of the tube.

Length of mercury thread at 15° C., .	15·43 cms.
Weight of mercury, ...	0·2134 gramme.

Take density of mercury at 15° C. as 13·59.

The weight of mercury which fills 1 cm. length of the tube is

$$\frac{0{\cdot}2134}{15{\cdot}43} \text{ grammes,}$$

and the volume of this mass at 15° C is

$$\frac{0{\cdot}2134}{15{\cdot}43 \times 13{\cdot}59} \text{ c.c.,}$$

or $$0{\cdot}00102 \text{ c.c.}$$

This is the capacity of 1 cm. length of the bore of the tube.

If r denote the radius of the bore in cms. then $(1 \times \pi r^2)$ is the capacity of 1 cm. length of the bore. We, therefore, have

$$\pi r^2 = 0{\cdot}00102,$$

or

$$r^2 = \frac{0{\cdot}00102}{\pi}.$$

That is,

$$2r = 2\sqrt{\frac{0{\cdot}00102}{\pi}} = 0{\cdot}018.$$

That is, the diameter of the bore of the tube is 0·018 cm.

A similar method based on the principle of Archimedes may be employed for finding the volume of a body. The body is weighed first in air, and then in water; the apparent loss of weight thus observed is the weight of the displaced water, and the volume of this water is exactly equal to the volume of the body by which it is displaced. The volume of the body can, therefore, be calculated from its apparent loss of weight in water and the density of the water at the temperature of weighing. Thus, if w denote the apparent loss of weight in water at t° C., and d the weight of unit volume of water at t° C., the volume of the body at t° C. is $\frac{w}{d}$ units of volume.

Experiment 29.—Find the diameter of a fine wire by the method described above.

Measure off, as exactly as possible, a metre of the wire, coil it up, and weigh it in air. Then wash it in dilute caustic soda and water, and weigh it in water. From the apparent loss of weight, and the density of the water, calculate out the diameter of the wire.

Example.—In an experiment of this kind the following data were obtained. Calculate the diameter of the wire.

Weight of 1 metre of wire in air, . .	1·8642 grammes.
Weight of wire in water, . . .	1·6537 ,,
Temperature of water, ...	15° C.

Density of water at 15° C. is 0·999 gramme per c.c.

The apparent loss of weight in water is here (1·8642 − 1·6537) grammes, or 0·2105 gramme.

The volume of water displaced by the wire at 15° C. is, therefore, $\frac{0{\cdot}2105}{0{\cdot}999}$ c.c., and this is also the volume of the wire at 15° C.

If r denote the radius of the wire in cms. the volume of the wire is $100 \times \pi r^2$ cub. cms. We, therefore, have

$$100\,\pi r^2 = \frac{0{\cdot}2105}{0{\cdot}999},$$

or

$$r^2 = \frac{0{\cdot}2105}{100 \times \pi \times 0{\cdot}999},$$

and

$$r = \sqrt{\frac{0{\cdot}2105}{100 \times \pi \times 0{\cdot}999}} = 0{\cdot}0259.$$

That is, the diameter of the wire is 0·0518 cm.

The methods illustrated by the examples given in this article are in very general use, and, when carefully carried out, they are probably the most accurate methods of measuring capacity or volume.

CHAPTER XX.

PROPERTIES OF LIQUIDS.

116. **Compressibility of Liquids.**—It has already been explained that although a liquid possesses no elasticity of form or rigidity, it possesses elasticity of volume in a marked degree. The volume elasticity of a liquid can, however, be exhibited, under ordinary conditions, only by the resistance it offers to compression, and the **compressibility** of a liquid is, therefore, the only elastic property by which the volume elasticity of the liquid can be determined experimentally.

The general method which has been adopted in studying the compressibility of a liquid has been to place the liquid in a glass tube made up of a bulb and a graduated stem, and then to subject the liquid to pressure, either by immersing the open tube in water under pressure, or by putting the interior of the tube in communication with a reservoir of compressed air. The compression of the liquid was then indicated by the graduations on the stem of the tube, and the pressure applied was measured by a suitably arranged manometer.

A tube of this kind was first used for this purpose by Oersted, and is usually known as a **piezometer**. One form of the tube is shown in Fig. 142; it is a glass tube of sufficient strength to resist considerable pressure, and consists, as shown, of a long cylindrical bulb and a long fine-bore stem graduated in divisions whose volume relative to the capacity of the tube up to the zero of the scale has been accurately determined by calibration with mercury.

One of the simplest and most satisfactory methods of making a determination of the compressibility of a liquid by means of the piezometer is that originally adopted by Regnault. The tube is filled to a suitable point in the graduated stem with the liquid whose compressibility is to be determined, and is then immersed in water to which great pressure can be applied. The water is contained in a vessel specially fitted for the application and measurement of the necessary pressure, and capable of withstanding the pressure so applied. The liquid in the open piezometer tube is in this way subjected to known pressure and suffers corresponding compression; at the same time, the tube is subjected to the same pressure internally and externally, and is, therefore, compressed to the same extent as a solid piece of glass of the same external volume would be compressed. It follows from this that the compression of the liquid, as indicated by its apparent change of volume in the piezometer tube, is really the difference between the real compression of the liquid and the compression of the tube, and that the real compressibility of the liquid can be found from the data of the experiment only if the compressibility of the material of the tube is known. This fact makes the accurate determination of the compressibility of a liquid very difficult, but satisfactory methods have been devised for finding the compressibility of the material tube, and the determination of liquid compressibilities can now be made with some accuracy.

Fig. 142.

It will be seen from what has been said above that if V denote the volume of liquid in the piezometer tube, v the *real* decrease of volume produced by compression under a hydrostatic pressure p, then v/V measures the volume *strain* produced in the liquid, and $\frac{pV}{v}$ is *the modulus of volume elasticity* for the liquid.

Instead of calculating out the modulus of volume elasticity in this way it is more usual to give what is called *the coefficient of compressibility* of the liquid. This coefficient is merely the decrease in volume per unit volume per atmosphere increase of pressure. That is, if p in the notation used above be taken to denote the increase of pressure in atmospheres, then $\frac{v}{Vp}$ is the coefficient of compressibility which is thus seen to be the reciprocal of the modulus of volume elasticity when p is measured in atmospheres.

The following coefficients of compressibility for water, sea-water, and mercury were given by Professor Tait, and serve to indicate the order of magnitude of this constant for liquids:—

Liquid.	Coefficient of Compressibility.
Water, ...	0·0000047
Sea water, ...	0·0000041
Mercury, ...	0·0000036

This result means, for example, in the case of water that an increase in pressure of one atmosphere will compress any given volume of water by a little less than five millionths of its initial volume.

117. **Viscosity.**—Although a liquid offers no elastic resistance to change of form, it is found that most liquids resist this change in some degree as the result of molecular friction between adjacent layers of the liquid. This resistance merely acts as an opposing force while the change of form is in progress, and does not tend at any stage in the process to restore the mass to its initial form. It is of zero value at any instant when the rate of change is zero, and it is found that its value at any instant while the change is in progress is proportional to the *rate* at which the change takes place.

The property of a liquid which enables it to offer this frictional resistance to change of form is called **viscosity.** Any liquid, such as glycerine, which possesses this property in a marked degree is called a *viscous liquid*, while a liquid such as water or alcohol, which is of comparatively low viscosity, is called a *mobile liquid*.

It will be seen from what has been said above that if a viscous substance changes form *very slowly* the resistance which it offers to the change will be *very small*. That is, a liquid substance, whatever may be its viscosity, will undergo change of form under the action of the smallest force; but, if the viscosity of the substance is great, the rate at which the change of form goes on may be very slow. If we adopt this fact as the criterion of a liquid it is found that a number of substances which appear to be solids are essentially liquids of very great viscosity. Thus, substances such as pitch, sealing-wax, cobbler's wax, and other substances are found to undergo continuous and progressive change of form under the influence of a small deforming force. A mass of pitch, for example, if placed on a table, will, in time *flow* over the surface of the table and cover it with a thin sheet of pitch. This flow takes place at constant temperature below the melting point, so that it is not due to melting, but the viscosity increases very rapidly as the temperature falls below the melting point. Substances which behave in this way are probably at the ordinary temperature in the semi-plastic state which precedes melting and which may extend, in some substances, over a considerable range below the melting point. The method of defining and measuring the viscosity of a liquid is based upon the following considerations.

Let ABCD, in Fig. 143, represent at a given instant a horizontal rectangular plate of liquid, of very small thickness, in a mass of liquid flowing horizontally in the direction of the arrow, and suppose the rate of flow in any horizontal layer to decrease with the depth of the layer below the surface of the liquid, and to be slightly greater, therefore, in the layer CD than in the layer AB. Now, imagine the flow to continue for one unit of time from the instant considered, and let ABCD, in Fig. 144, represent the form of the plate at the end of this unit of time. Since the rate of flow in the layer CD is greater than in the layer AB, the upper face CD of the plate advances in unit time through a slightly greater distance than the lower face AB and the plate thus becomes *sheared* into the form shown in the figure. The shear thus produced in the plate *in unit time*

Fig. 143.

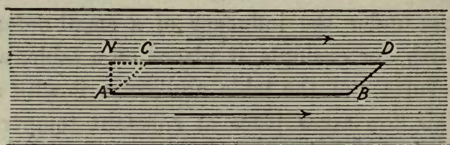


Fig. 144.

is evidently measured, as indicated in the figure, by the ratio $\frac{CN}{AN}$. That is, if v denote the *difference* in the velocities of the upper and lower faces of the plate, and d the thickness of the plate, the *rate* at which the plate is being sheared is measured by $\frac{v}{d}$.

Now, the stress to which this continuous shearing is due is the friction exerted by the adjacent liquid on the upper and lower faces of the liquid plate. The liquid layer immediately above the upper face flows a little more quickly than that face, and the viscous friction between the two liquid surfaces acts, therefore, on the surface of the plate in the direction of the flow. Similarly, the liquid layer immediately below the lower face flows a little more slowly than that face, and the viscous friction between the liquid surfaces acts, therefore, on the surface of the plate in a direction opposite to the direction of flow. The liquid plate is thus sheared by the opposing frictional stresses on its upper and lower faces; these stresses act parallel to the faces of the plate, and if the plate is very thin then the liquid friction per unit area of surface may be taken to be the same for both faces.

If this frictional stress on the faces of the liquid plate be denoted by F, the ratio of the stress, F, to $\frac{v}{d}$, the shear produced in the plate *per unit time* is taken as a measure of the *viscosity* of the liquid, as is called *the coefficient of viscosity* for the liquid. That is, if m denote the viscosity of the liquid, we have

$$m = \mathrm{F} \Big/ \frac{v}{d} = \frac{\mathrm{F}d}{v}.$$

It is to be noted that *the stress* F *exists only while the shearing is going on in the liquid*, so that m must be defined in relation to the shearing strain *per unit time.**

When a liquid which wets glass flows through a glass tube of fine bore a stationary film of liquid adheres to the tube, and the rate of flow, which is zero for this film, increases from zero at the outer surface of the stream to a maximum at the axis of the stream. It follows from this that any infinitely thin cylindrical shell of the liquid stream flows a little more rapidlv than the

* In comparing m, the coefficient of viscosity for a liquid, with n, the modulus of simple rigidity for a solid, it is to be noted that the frictional stress between the layers of the liquid *while shearing is going on continuously* corresponds to the elastic stress set up between the layers of the solid *in opposition to the existence of any given shear.* Hence, in defining m, the shear produced *per unit time* must be taken for the shearing strain; whereas, in defining n, only the magnitude of the shear produced by the applied stress has to be considered.

similar shell immediately surrounding it, and that there must, therefore, be viscous friction between the two shells of liquid. That is, the flow of the liquid within any cylinder taken inside the bore, and co-axial with it, is retarded by the viscous friction of the liquid surrounding it, and the rate of flow of liquid through the tube must, therefore, depend upon the viscosity of the liquid. It can, in fact, be shown that the rate of flow under given conditions through a given tube is *inversely proportional* to the viscosity of the liquid, as defined above.

The viscosity of different liquids may, therefore, be compared by comparing the times in which the same quantity of each liquid flows through the same capillary tube under the same conditions of flow.

It is a matter of common observation that a viscous fluid such as glycerine becomes less viscous with rise of temperature, and experiment shows that it is generally true for all liquids that viscosity decreases as the temperature rises.

Approximate values of the coefficients of viscosity at 0° C. and 20° C. are given below for a few typical liquids :—

Liquid.	Coefficient of Viscosity.	
	At 0° C.	At 20° C.
Water, ...	0·0181	0·0102
Alcohol, ...	0·0018	0·0013
Ether, ...	0·0030	0·0026
Mercury, ...	0·0170	0·0150
Glycerine, ...	40·0000 (At 4° C.)	5·0000

CHAPTER XXI.

PROPERTIES OF LIQUIDS *(Continued).*

118. **Surface Tension and Capillarity.**—Certain phenomena, which may be observed at the boundary surface of a mass of liquid, seem to indicate, and may be explained by assuming, that the thin surface layer or skin which forms the boundary surface of the liquid acts as a stretched membrane under uniform tension in all directions.

Thus, if a large drop of oil is allowed to form on the surface of water, it rests on the water as a lens-shaped mass, and the surface of separation between it and the water is distinct and clearly marked, just as it would be if each mass of liquid were contained in a very thin elastic skin or membrane. In the same way if a light object, such as a needle, is slightly greased and placed very carefully on the surface of water, it will be supported by the surface, just as it would be by a thin-stretched membrane. The surface shows a slight depression at the point where the object rests, and the weight of the object may be supposed to be supported by the vertical components of the surface tension round the edge of the depression.

When a small quantity of mercury is spilt on the table or on the floor, it usually breaks up into a large number of small globules, which are nearly spherical in form. This is readily explained by supposing that the outer surface layer of each globule is subject to uniform tension, and that it, therefore, assumes a spherical form in which its surface area is a minimum. The same effect may be exhibited in a striking way by one of the many beautiful experiments due to **Plateau.** If a quantity of olive oil is passed gently from a pipette into a mixture of water and alcohol of the same density as the oil, the effect of gravity in preventing the formation of large drops is eliminated, and the oil takes the form of a large spherical drop, separated by a clearly marked boundary surface from the surrounding liquid. A similar effect may be observed in the formation of drops at the end of a glass rod or pipette which has been dipped into a liquid. The forms which the drop assumes during its formation, and the manner in which it breaks away from the rod to which it is attached, are all consistent with the supposition that the surface layer of the drop acts as a stretched membrane under uniform tension in every direction.

The fact that the surface of a liquid behaves as if subject to tension is further illustrated by the behaviour of liquid films. When a soap bubble is blown from soap solution, the double film which forms the wall of the bubble is extended by increasing the pressure inside the bubble, and if this pressure is removed the wall of the bubble contracts and drives out the air which has been forced into it. That is, the bubble behaves as if it were a very thin elastic membrane under uniform tension.

Numerous other experiments show that a liquid film always behaves in this way. When a flat wire ring is dropped into soap solution and withdrawn, a thin film of the solution will be found stretched across it. If a loop of thread which has been dipped in the solution is then placed gently on this film it will retain any irregular form that may be given to it, but if the film is broken at a point within the loop the portion of the film between the wire ring and the loop at once contracts, and the loop takes a circular form as the inner boundary of the film.

The tension which is thus supposed to exist in a very thin layer or skin at any boundary surface of a liquid is called the **surface tension** of the liquid. It is supposed to have the same value for all directions in the surface, and is measured for any surface by the tension at right angles across unit length of a line taken in any direction on the surface of the liquid. That is, the surface tension is the tension in a strip of the surface film of unit width, taken in any direction on the surface.

The magnitude of the surface tension at the boundary surface of a liquid is found to depend upon the substance from which the liquid is separated by the surface. Thus, the surface tension of water at the boundary surface between water and air differs from the surface tension of water at the boundary surface between water and oil, or between water and glass.

It must be understood, too, that the surface tension of a liquid for any boundary surface is not increased by extending the surface. The surface film of a liquid cannot be *stretched* in the sense that an elastic membrane is stretched; it may be *extended* by the transfer of molecules from the deeper layers into the surface layer, but the tension in the surface remains *constant* during the process of extension. Thus, in the case of a soap bubble, the wall of the bubble consists of an inner or outer surface film enclosing a very thin layer of liquid between them; as the bubble is blown the inner and outer films are extended at the expense of the liquid between them until all their liquid is transferred to the surface films and the bubble bursts. The films thus remain in the same state and subject to the same tension throughout the process of extension.

It will be seen from what has been said that if a strip of liquid surface of width (a) is extended *in length* by a distance (b), the work done during the process of extension is given by $T(ab)$, where T denotes the surface tension for the liquid surface. That is, the work done in extending a liquid surface by an area (ab) is $T(ab)$, or, in general, the work done in extending a liquid surface by an area A is TA, where T denotes the surface tension of the surface. It follows from this that the energy of a liquid surface *per unit of area* is numerically equal to its surface tension, and that the energy which a liquid possesses as surface energy increases when the surface area is extended, and decreases when the surface area is diminished. The energy which a liquid surface possesses in virtue of its surface tension is called the **surface energy** of the surface.

The interesting phenomena which are usually considered under the head of **capillarity** or **capillary action** are due also to surface tension. When a rod or plate of any solid is dipped into a liquid **which does not wet the solid,** it is noticed that the liquid surface is convex and slightly depressed round the line of contact with the solid. Thus, if a rod or strip of glass is dipped into mercury, the liquid surface round the line of contact takes the form shown in section in Fig. 145. If the mercury is

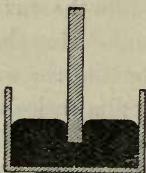


Fig. 145.

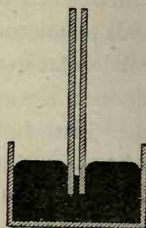


Fig. 146.

contained in a glass vessel the same effect may also be observed round the line of contact of the mercury surface with the glass.

Further, if one end of a capillary glass tube, about 1 mm. or less in bore, and open at both ends, is dipped into mercury so that a column of mercury enters the bore, it is found that the surface of the mercury column is not only convex, owing to the convex curvature round the line of contact with the glass, but that it stands at a lower level than the general surface of the mercury outside the tube. This effect is shown in section in Fig. 146; it will be seen that the mercury surface is convex and slightly depressed round the lines of contact of the surface with the glass inside the tube, outside the tube, and round the inner surface of the containing vessel, and also that the level of the mercury inside the tube is slightly lower than the general level outside. If tubes of different bores are used in this experiment it is found, too, that the *capillary depression* inside the tube increases as the bore diminishes, and measurement shows that it is inversely proportional to the diameter of the bore of the tube.

Experiment 30.—Take three or four lengths of capillary glass tubing of different bores, place them in succession with one end dipping in mercury, and measure the capillary depression in each case. This may be done by bringing the tube close to the side of the glass vessel containing the mercury ; the depression of the thread of mercury in the tube can then be measured directly with a scale outside the tube, or read off on a scale attached to the tubing.

Measure also the diameter of the bore of each tube and establish the relation between the capillary depression and the diameter.

Again, when a rod or plate of a solid is dipped into a liquid **which wets the solid,** it is seen that the liquid surface is con-

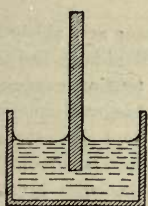


Fig. 147.

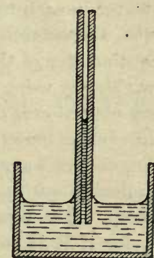


Fig. 148.

cave and slightly elevated round the line of contact with the solid. Thus, if a rod or strip of clean glass is dipped into water, the liquid surface round the line of contact takes the form shown in section in Fig. 147. The same concave curvature and elevation of the surface may also be observed round the line of contact of the water with the vessel containing it.

In the same way if one end of a capillary glass tube open at both ends is dipped into water, or any other liquid which wets glass, so that liquid enters the bore, it is found that the surface of the liquid in the tube is not only concave, owing to the concave curvature round the line of contact with the inner wall of the tube, but that it stands, as shown in Fig. 148, at a higher level than the general level of the liquid outside the tube. If tubes of different bores are tried it is found that the *capillary elevation* in the tube increases as the bore diminishes, and measurement shows that it is inversely proportional to the diameter of the bore. It is found also, as might be expected, that the capillary elevation in a tube of given bore is different for different liquids.

Experiment 31.—Take three or four lengths of capillary glass tubing of different bores, attach each in turn with small rubber bands to a thermometer stem, or a strip of glass or porcelain with a millimetre scale engraved on it, and place it standing vertically with one end dipping in water. Then note and measure the capillary elevation of the water in each tube. Measure also the diameter of the bore of each tube, and establish the relation between the capillary elevation and the diameter of the bore.

Show also, by using different liquids (water and alcohol), that the capillary elevation is different for different liquids in the same tube.

In these experiments, and in all surface tension experiments, it is necessary to make sure that the tubes, liquids, and containing vessel are chemically clean to begin with, and are kept clean throughout the experiment. The liquid, for example, must not be touched with the fingers, for if it is so touched the surface becomes greasy, and its surface tension alters in magnitude.

The phenomena due to surface tension in **capillary tubes** and other similar and related phenomena constitute what is known as **capillarity** or **capillary action.**

The explanation of the manner in which capillary depression and capillary elevation are associated with surface tension may be given in the following way.

Consider first the case in which the liquid does not wet the solid, and in which the liquid surface is convex and depressed along the line of contact of the liquid and the solid. It will be seen that the line of contact of the liquid and the solid is really the boundary line between the liquid surface in contact with the air, and the liquid surface in contact with the solid. Thus, in Fig. 149, the surface of the liquid is in contact with the solid along DA, and in contact with air along ABC, and the point A is evidently a point on the boundary line between these two surfaces. It may also be assumed that the surface tension in the liquid-solid surface DA will differ from the surface tension in the liquid-air surface ABC. It follows, therefore, as a condition of equilibrium of the surface at A, that the two surfaces DA and ABC must meet at an angle, and if the surface tension of the liquid-solid surface, DA, is less than the surface tension of the liquid-air surface, it will be seen that this angle, DAE, must be an obtuse angle as in the figure. For, if T denote the surface tension of the surface ABC, and T′ the surface tension of the surface DA, it is evident from the figure that for equilibrium in a vertical direction at A we must have

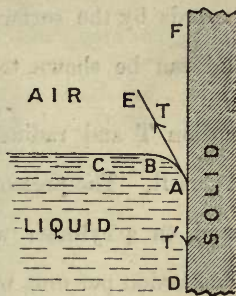


Fig. 149.

$$T \cos a = T',$$

where a denotes the angle EAF, or the supplement to the angle DAE, at which the two surfaces meet. This relation shows that if T′ is less than T, the angle DAE must be obtuse, and that the surface ABC must necessarily assume a convex curvature in joining the surface DA at this angle.

This angle, DAE, is known as **the angle of contact** of the liquid with the solid, although it might be more pertinently defined as the angle of equilibrium for the two liquid surfaces meeting at the point A.

The depression of the liquid below the general level at all points where the surface is curved, is explained by the fact that in the case of a surface under tension the pressure on the concave side of the surface is greater than the pressure on the convex side by an amount which depends on the tension and curvature of the surface. This excess of pressure on the concave side of a surface under tension is the pressure exerted inwards by the surface in virtue of its tension and curvature, and can be shown to be equal to $\frac{T}{R}$ for a *cylindrical surface* of tension T and radius R, and to $\frac{2T}{R}$ for a *spherical surface* of radius R. The pressure exerted, for example, by a flat elastic band on a cylinder which it encircles is $\frac{T}{R}$, where T denotes the tension per unit width of the band, and R the radius of the cylinder. Similarly, the pressure of the air inside a soap bubble is $2\left(\frac{2T}{R}\right)$ greater than the external pressure, where T denotes the surface tension of the liquid films, and R the radius of the bubble.

The pressure at any point in the liquid immediately below the convex surface ABC (Fig. 149), is thus greater than the atmospheric pressure at a point immediately above the surface, and this excess of pressure at any point in the surface compensates exactly for the loss of hydrostatic pressure due to the capillary depression of the surface at that point.

This result may readily be applied to determine the relation between the surface tension of a liquid, and the capillary depression of that liquid in a capillary tube of material which is not wetted by the liquid. Thus, if the capillary depression in a capillary tube of radius r is denoted by h, and if the radius of curvature of the approximately spherical convex surface of the column of liquid in the tube is denoted by R, we have $\frac{2T}{R}$ as the pressure due to the tension and curvature of this surface, and *hdg*, where *d* denotes the density of the liquid, as the loss of hydrostatic pressure due to the capillary depression. We may, therefore, write—

$$\frac{2T}{R} = hdg.$$

But, if a denote the angle of contact of the liquid with the

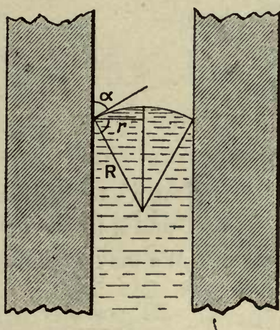


Fig. 150.

solid, and r the radius of the bore of the tube, it can readily be seen from Fig. 150 that $\frac{r}{R} = \cos a$, or $R = \frac{r}{\cos a}$. We, therefore, have—

$$\frac{2T \cos a}{r} = hdg,$$

or

$$h = \frac{2T \cos a}{rdg}.$$

This result shows, as stated above, that the capillary depression, h, for a given liquid, is inversely proportional to radius of the bore of the tube; it also shows that the value of h

is different for different liquids, and is, for a tube of given bore and material, directly proportional to T, the surface tension of the liquid-air surface, and inversely proportional to d, the density of the liquid.

This result may be obtained, perhaps more simply, by supposing the column of liquid in the tube to be held down by the tension in the cylindrical liquid-solid surface acting downwards round the edge of the convex liquid-air surface which covers the upper end of the column. For, if T′ denote the surface tension of this surface, we have—

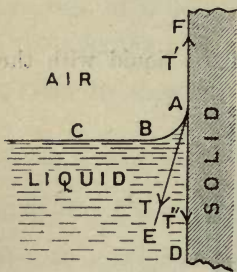


Fig. 151.

$$2\pi r T' = \pi r^2 hdg,$$
$$2T' = rhdg,$$
or
$$h = \frac{2T'}{rdg}.$$

Now T′, as explained above, is related to T and a in the foregoing notation by the relation $T' = T \cos a$. We may, therefore, write—

$$h = \frac{2T \cos a}{rdg},$$

as given above.

Consider next the case in which the liquid wets the solid, and in which the liquid surface is concave and elevated along the line of contact of the liquid and the solid. In this case it will be seen, as shown in Fig. 151, that the angle of contact DAE is acute, and that, as a condition of equilibrium at the point A in the liquid, the force exerted on a molecule at A by the surface films in contact with the wall of the tube along DAF must act vertically upwards. This condition will be realised, assuming that, since the liquid wets the tube, and since the air in the tube above the liquid column is saturated with the vapour of the liquid, a surface film must extend up the tube to a considerable height above the point A. This film separates the vapour-saturated air in the tube from the solid wall, and may be called the air-solid surface. Then, if T′ denote the tension in this air-solid surface acting along AF, and T″ the tension in the liquid-solid surface acting along AD, the force exerted on a molecule at A by the surface films in contact with the wall of the tube along DAF will be T′ − T″, acting vertically upwards if T′ be supposed to be greater than T″.

If these assumptions be made, and if T denote the surface tension of the liquid-air surface ABC, and a the angle of contact, we have, as the condition of equilibrium in a vertical direction at A, that

$$T \cos a = T' - T''.$$

When the liquid wets the tube the value of a is always very small and is generally taken to be of zero value.

The capillary elevation which accompanies the concave curvature of the liquid surface along the line of contact of the liquid and the solid when the liquid wets the solid, is explained in exactly the same way as the capillary depression which accompanies the convex curvature of the surface along the line of contact when the solid is not wetted by the liquid. Thus, at any point in the concave surface ABC (Fig. 151), the pressure immediately below the surface is less than the atmospheric pressure immediately above the surface by an amount $\frac{2T}{R}$, where T denotes the surface tension of the surface and R the radius of curvature at the point considered, and this difference of pressure at any point in the surface compensates exactly for the loss of hydrostatic pressure due to the capillary depression of the surface at that point.

The relation between the surface tension of a liquid and the capillary elevation of that liquid in a capillary tube of material which is wetted by the liquid, may also be determined by applying this result as in the case of the capillary depression of a liquid in a capillary tube which is not wetted by the liquid. Thus, if the capillary elevation of the liquid in a tube of radius r is denoted by h, and if the radius of curvature of the upper surface of the volume of liquid in the tube is denoted by R, we get $\frac{2T}{R}$ as the pressure due to the tension and curvature of the surface, and hdg, where d denotes the density of the liquid, as the gain of hydrostatic pressure due to the capillary elevation. We, therefore, have

$$\frac{2T}{R} = hdg,$$

and $R = \frac{r}{\cos a}$, as explained above, where a denotes the angle of contact, we have

$$\frac{2T \cos a}{r} = hdg,$$

or

$$h = \frac{2T \cos a}{rdg}.$$

If we now assume $a = 0$ we get

$$h = \frac{2T}{rdg}.$$

This result is identical with that obtained above for capillary depression. It shows that h is inversely proportional to r, the radius of the bore of the tube, directly proportional to T, the surface tension of the liquid-air surface, and inversely proportional to d, the density of the liquid.

The result may also be obtained by supposing the column of liquid to be held up in the tube by the attachment of the concave surface on its free upper surface to the wall of the tube. For if T denote the surface tension of this surface we have—

$$2\pi r\mathrm{T} \cos a = \pi r^2 hdg,$$
$$2\mathrm{T} \cos a = rhdg,$$
$$h = \frac{2\mathrm{T} \cos a}{rdg},$$

or, if $a = 0$ we get

$$h = \frac{2\mathrm{T}}{rdg}$$

as given above.

The surface condition here dealt with as surface tension is generally considered to be the result of the molecular forces acting on the molecules of the liquid at the boundary surface of the liquid.

A molecule in the interior of the liquid is subject to the action of the molecules surrounding it within a certain small range, and as it is thus acted on equally in all directions the resultant molecular force acting on it is of zero magnitude. A molecule at or near the boundary surface of the liquid is subject, however, on one side, to the action of the molecules of the liquid within molecular-range, and on the other side to the molecules of the material adjacent to the liquid surface; the resultant molecular force acting on the molecule is, therefore, not in general of zero value, and the constraint thus imposed on the molecules in a thin surface layer of the liquid constitutes what is called surface tension.

119. **Diffusion in Liquids.**—When a solution of a salt in any solvent is not of uniform concentration throughout its mass molecules of the dissolved substance pass from points of greater to points of less concentration, and molecules of the solvent pass from points of less to points of greater concentration until the solution is of uniform concentration at all points.

This systematic migration of the molecules from point to point in the liquid is not accompanied by any perceptible motion in the liquid, and takes place in any direction quite irrespective of the action of gravity.

For example, if a concentrated solution of copper sulphate in water is placed in a beaker, and water is poured very gently down the sides of the beaker on the surface of the solution, the horizontal surface of separation between the water and the solution is at first quite sharp and distinct, but after a time molecules of the dissolved salt pass upwards into the water, and molecules of water pass downwards into the solution. The blue colour, indicating the presence of copper sulphate in solution, thus extends upwards into the water, becoming fainter and fainter as the distance above the initial plane of separation increases, and the blue colour of the concentrated solution becomes fainter as it is diluted by the water which passes down into it.

This process of mixing by molecular motion without the aid of currents in the solution is known as **diffusion.**

The process is a very slow and gradual one in liquids, unless assisted by mechanical mixing or stirring. In the case just considered, for example, the process might take many weeks or even years (according to the depth of the layers of liquid) to reach completion.

Diffusion in liquids was first studied experimentally by Graham. He was able to compare the diffusivity of different salts in water as a solvent by filling a wide-mouthed bottle with a solution of the salt to be tested, and then placing the bottle to stand on the bottom of a large beaker full of water. The salt diffused from the solution in the bottle into the surrounding water, and the quantity which thus diffused out of the bottle in a given time was determined for a number of different salts and for different concentrations of the same salt. Graham found that for solutions of the same concentration the rate of diffusion was different for different substances, but that for solutions of the same substance the rate was proportional to the concentration.

The quantitative study of diffusion was not, however, put on a satisfactory basis until the general method due to Fourier* was applied to the process by Fick. By this method a definite coefficient of diffusion is defined, and the value of this coefficient under given conditions can be determined experimentally for any substance.

Imagine a solution in which diffusion is going on to be arranged in plane parallel layers in which the concentration† is the same at all points in any one layer, but differs from layer to layer, and let δ denote the difference in concentration for any two adjacent layers a very small distance x apart. The *gradient of concentration* from one layer to the other will thus be denoted by $\frac{\delta}{x}$, and the quantity of the dissolved substance, which diffuses across unit area in unit time from layer to layer, is taken to be proportional to this gradient, and is, therefore, denoted by $k\frac{\delta}{x}$, where k is a constant. This constant, k, is the **coefficient of diffusion** for the dissolved substance. Experiment shows, however, that the value of k is not strictly a constant for any substance, but that it varies to some extent with the concentration of the solution.

The general method of determining k experimentally for any substance cannot here be explained. It may be noticed, however, that in order to make a determination it is necessary to arrange that the diffusion shall take place between parallel plane layers of uniform density; and, also, to be able to note, quantitatively, the changes of concentration that take place in the different layers as the process of diffusion goes on. This has been done in various ways; the simplest method of experiment is that adopted by Lord Kelvin. The solution was

* This method is the one adopted in defining the absolute thermal conductivity of a substance. (See Part iv. on *Heat.*)

† The *concentration* may, for this purpose, be defined as the mass of the dissolved substance per unit volume of the solution.

placed at the bottom of a tall cylindrical vessel and allowed to diffuse upwards through an overlying layer of the solvent, as explained above. In order to follow the process of diffusion a number of small beads, constructed to float in solutions of known densities intermediate between the density of the solvent and the initial density of the solution, were placed in the vessel. At first these beads floated all together at the surface of separation of the solvent and the solution, but as diffusion went on the lighter beads rose, and the heavier beads sank, each bead taking up its position in a layer of the density it was constructed to indicate. The density, and therefore the concentration in layers of known position, could thus be observed from time to time during the progress of the diffusion, and the data necessary for the determination of the coefficient of diffusion could be deduced from these observations. It was found that the value of the coefficient for a given substance depends upon the solvent used and also, to some extent, upon the concentration of the solution.

120. **Osmosis.**—Graham found in his experiments on diffusion in liquids that certain substances, such as mineral acids and salts, diffused more or less readily; whilst other substances, such as gum, starch, gelatin, and other similar substances possessed little or no power of diffusion in solution. The former class of substances he called *crystalloids*, because they are, in most cases, capable of crystallisation, and the latter, which are incapable of crystallisation, he called *colloids*.

Graham found, also, that a colloid substance in combination with a very small quantity of water readily forms a solid or semi-solid jelly through which water and crystalloid substances in solution readily pass by diffusion, but which is quite impervious to other colloid substances. He found, too, that an animal membrane, such as a piece of bladder, and the "parchment" obtained by treating paper or cellulose with sulphuric acid behaved like a thin film of colloid jelly, and while it allowed water and other liquids and crystalloids in solution to pass through it freely, it was quite impervious to all colloid substances.

This passage of liquids and substances in solution through "colloid" films or membranes is called **osmosis.** Graham applied the process as a means of separating a crystalloid from a colloid substance. The mixture of the two substances in solution is placed in a cylindrical vessel, the bottom of which is formed by a sheet of parchment paper, and the vessel is immersed to about half its depth in water; the crystalloid then diffuses through the parchment into the water, and the colloid is left behind in the vessel. This method of separating a crystalloid from a colloid by osmosis is known as *dialysis.*

It has been found that certain membranes through which osmosis can take place exercise a kind of selective power in their action, and are permeable to certain substances, but quite impermeable to others. A piece of bladder, for example, is permeable to water, but impermeable to alcohol, so that if a bladder containing alcohol is immersed in water, the water enters the bladder and gradually distends it to bursting point.

A very important case of this selective action of the membrane is that in which the membrane is permeable to the solvent, but impermeable to the dissolved substance in certain solutions. A membrane possessing this property is said to be **semi-permeable,** and it is found that certain vegetable cell walls and certain films which can be prepared artificially do possess this property in relation to certain solutions. Thus, a film of copper ferrocyanide forms a semi-permeable membrane of this kind for certain solutions, such as a solution of sugar. If a porous pot is filled with a solution of copper sulphate and immersed in a solution of potassium ferrocyanide the two solutions penetrate the wall of the pot, and at all points where they meet in the thickness of a wall thin films of copper ferrocyanide are formed across the narrow channels which penetrate the wall.

Although these films are very thin and delicate they form, when deposited in this way in the wall of a porous vessel, a practically continuous membrane capable of withstanding considerable pressure. The solutions used in preparing the membrane are removed by soaking the vessel in water, and, when prepared in this way, the vessel is found to act effectively as a continuous semi-permeable membrane.

If a solution of sugar in water is placed in a porous vessel prepared in this way, and the vessel is immersed to half its depth in water, it is found that water passes freely into the solution, but that no sugar passes out into the water. Further, if the vessel is filled with the solution and closed by a rubber stopper or well paraffined cork fitted with a vertical tube, as shown in Fig. 152, it is found that as water enters the solution through the semi-permeable wall of the vessel the solution rises in the vertical tube, and that water continues to enter even against the back pressure due to the column of solution in the tube. Experiment shows, however, that when this back pressure reaches a certain value, which depends for a given solution upon the concentration of the solution, the water ceases to enter, and equilibrium is established between the solution inside the vessel and the water outside. The value of the back pressure at which this state of equilibrium is reached is called the **osmotic pressure** of the solution, and is considered to be the measure of the molecular pressure exerted by the dissolved substance in the solution.

Fig. 152.

CHAPTER XXII.

PROPERTIES OF GASES.

121. **Measurement of the Pressure of a Gas.**—It has been explained previously that the pressure of a gas is the same at all points in the gas, and the same in all directions at every point. It has also been explained that the pressure is to be measured by the pressure exerted by the gas per unit of area of any surface with which it is in contact.

In practice the pressure of a gas is generally measured by observing the height of the column of liquid (usually mercury), which exerts a pressure equal to that of the gas. The pressure of a gas in the flask F may, for example, be measured by connecting it, as shown in Fig. 153, with the U-tube M, which is about half filled with mercury or some other suitable liquid. The air in the flask exerts pressure on the surface of the mercury at A, and the atmosphere presses on the surface at B. If the pressure of the air in the flask is equal to the atmospheric pressure, the surfaces of the mercury at A and B will be at the same level as in Fig. 153; if, however, the pressure of the air is greater than the atmospheric pressure, the mercury will be forced outwards from the flask until a balance is obtained, and the level at B will be higher than the level at A as in Fig. 154; similarly, if the pressure of the air is less than the atmospheric pressure, the mercury will be forced inwards until a balance is obtained, and the level at B will be lower than the level at A as in Fig. 155.

In Fig. 154 it will be seen, as explained in Art. 113, that the pressure on the mercury surface at A is equal to the pressure in the other limb of the tube at a point A′ at the same level as A, and that the pressure of the air in the flask is, therefore, equal to the atmospheric pressure at B *plus* the pressure due to the mercury column A′B. Similarly, in Fig. 155 the atmospheric pressure on the mercury surface at B is equal to the pressure in the other limb at a point B′ at the same level as B, and that

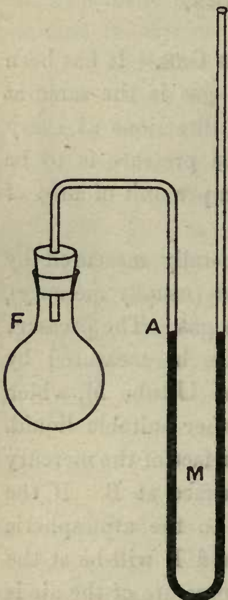


Fig. 153.

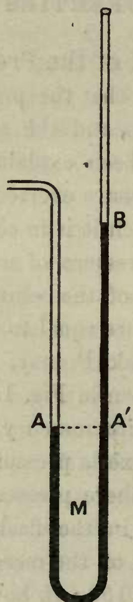


Fig. 154.

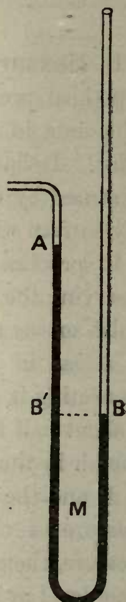


Fig. 155.

the pressure of the air in the flask is, therefore, equal to the atmospheric pressure *minus* the pressure due to the mercury column AB′.

It thus appears that the pressure of the air in the flask is, in the general case, equal to the atmospheric pressure, *plus* or *minus* the pressure due to a column of mercury whose length is equal to the difference in the levels of the mercury at A and B. That is, if P denote the atmospheric pressure, p the pressure due to the column of mercury of height equal to the difference of the levels at A and B, and P′ the pressure of the air in the flask, then $P' = P \pm p$. If the atmospheric pressure P is known in dynes per square centimetre, the pressure denoted by p must be similarly expressed by means of the relation hdg (Art. 108), where h is the height of the column in cms., d the density of the mercury in grammes per cub. cm., and g the acceleration due to gravity at the place of observation.

If the atmospheric pressure is known to be equal to the pressure due to a column of mercury of height H, and the difference of the mercury levels at A and B is denoted by h, then the pressure of the air in the flask is conveniently expressed as that due to a column of mercury of height $(H \pm h)$ as the case may be.

It must be noted, however, that if the pressure of a gas is expressed in terms of the height of a column of mercury (or other liquid), the density of the mercury must be specified, usually by giving its temperature, and the value of g, the acceleration due to gravity at the place of observation, must be known.

A **U**-tube filled with mercury, and constructed for the purpose of measuring pressure in the manner indicated in Fig. 153, is generally called a **manometer** or **pressure gauge.**

122. **The Atmospheric Pressure: The Barometer.**—The atmosphere surrounds the earth as a spherical layer of air which extends, as an appreciable atmosphere, to a height of from two hundred to three hundred miles above the surface of the earth. We thus live at the surface of the earth at the bottom of a deep sea of air, and subject to the pressure produced in the air surrounding us by the weight of the overlying air. This pressure in the air around us is called the **atmospheric pressure.**

We are not sensible of the pressure existing in the air around us, because, at every point, it is exerted equally in all directions.

That is, the air exerts pressure on any surface with which it is in contact in a direction at right angles to the surface, and the magnitude of this pressure at any point is the same for all positions of the surface. It follows from this, as in Art. 109, by the principle of Archimedes, that the resultant effect of the air pressure on a body completely surrounded by air is an upthrust equal to the weight of the displaced air, and acting vertically upwards on the body through the centre of gravity of the displaced air. At the same time, the body being subject to a normal pressure at all points on its surface, is compressed to an extent which depends upon its elasticity of volume. Hence, if the hand is held out in the air with the palm horizontal, it is not forced downwards by the downward air pressure on the upper surface, or upwards by the upward pressure on the lower surface, but experiences merely a very small upward thrust equal to the weight of the air which it displaces. At the same time the tissue and blood-vessels of the hand are subject to the compressing or supporting effect of the pressure to which it is exposed. Both these effects are inseparable from the conditions of life at the surface of the earth, and we are not sensible of them because we are always subject to them. The upthrust due to the buoyancy of the air is too small to have any special relation to the structure of the body, but the supporting effect of the air pressure on the tissues of the body is one of the conditions to which the structure of the body is specially adapted, and which cannot be altered without danger to life.

In the same way if a thin glass flask is exposed to the air it is not crushed by the pressure of the air on it. The outward pressure of the air inside it on its inner surface practically balances the inward pressure of the external air on the outer surface, and the final result is merely that the glass of the flask is very slightly compressed, and that its pressure on the table on which it stands is less than its true weight by the weight of the air which it displaces.

The fact that the air around us does exert a very considerable pressure on all surfaces with which it is in contact may be demonstrated in a simple and striking manner by the familiar experiment described below. The experiments with an air pump, described in Arts. 128, 129, also illustrate this fact in an interesting way.

Experiment 32.—Get a cylindrical flask made of thin tin plate for the purpose of this experiment. Put a small quantity of water in the flask, and boil the water until all the air in the flask is expelled by the steam. Then, while the water is still boiling, cork up the flask with a good, well-fitting cork, and stand it in a suitable trough to cool it by pouring cold water over it.

As the flask cools, the steam inside it condenses, and a partial vacuum is produced inside the flask. The walls of the flask are now subject to the atmospheric pressure externally, and to a pressure diminished almost to zero value internally. They are not strong enough to withstand this excess of external pressure, and are consequently crushed violently inwards soon after the cooling begins.

A thin glass flask may be broken in the same way, or, if the flask is stout enough to stand the pressure, it will be found that the cork is forced inwards so as to fit very tightly into the neck of the flask.

The pressure in the air at any level being due to the weight of the air above this level must evidently be greatest at the surface of the earth, and must decrease from layer to layer as the height above the surface increases.

It will be understood from this that, as the air is a gas, and easily compressible under pressure, the density of the air is greatest at the surface of the earth, where the pressure is greatest, and decreases as the height above the surface increases, and the pressure diminishes until, at a height of from two hundred to three hundred miles above the sea level, the density and pressure become inappreciably small.

It follows from this that the pressure of the air at any level could not be found by applying the relation $p = hdg$, as explained in Art. 108, even if the value of h were definite and accurately known, for the value of d varies from layer to layer, and is, therefore, not a constant. The value of g, also, varies from layer to layer, for it decreases as the height above the sea level increases. For the same reasons it will be seen that the *difference of pressure* for a vertical *difference of level* denoted by h is not given by hdg. If, however, h is small, so that both d and g may be considered to have practically constant values at all points in it, this formula may be applied to obtain the difference of pressure corresponding to the small difference of level considered.

Example.—Find the difference in the pressure of the air at two points whose vertical difference of level is 10 metres, taking the mean density of the air between the points to be 0·0012 gramme per cub. cm., and the value of g to be 980 cms. per sec. per sec.

Here, if p denote the difference in the air pressures at the two points we have

$$p = hdg,$$

and, from the given data, the values of h, d, and g in C.G.S. units are

$$h = 100,\ d = 0{\cdot}0012, \text{ and } g = 980.$$

We, therefore, have

$$p = 100 \times 0{\cdot}0012 \times 980,$$

or

$$p = 117{\cdot}6.$$

That is, the difference in the pressure of the air at the two points is 117·6 dynes per sq. cm.

The existence of the atmospheric pressure, and the manner in which it may be measured at any point in the air, is indicated by the following historical experiment which is due to **Toricelli**, and is generally known as **Toricelli's experiment.**

A long glass tube, closed at one end, about a metre long and a centimetre in diameter, is filled with clean dry mercury and inverted in a small cistern of mercury. When thus inverted and fixed in a vertical position, in the manner shown in Fig. 156, it is found that the mercury in the tube falls until the height of the column in the tube is about 76 cms. above the level of the mercury in the cistern. The pressure due to this column of mercury evidently measures the atmospheric pressure at the surface of the mercury in the cistern. The pressure at any point A, *inside the tube* at the level of the surface of the mercury in the cistern, is the same as at any point *at the same level outside the tube*, and is, therefore, equal to the atmospheric pressure at the surface of the mercury. The pressure at the point A is, however, equal also to the pressure in the space BC *plus* the pressure due to the column AB of mercury. Now, if the experiment has been properly performed, the space BC above the column of mercury in the tube is devoid of air or other matter, and is, therefore, a vacuum in which the pressure is of zero value.* It follows, therefore, that the atmospheric pressure at the surface of the mercury in the cistern is equal to the pressure due to the column of mercury AB, which stands in the tube.

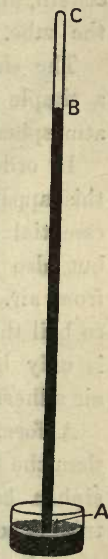


Fig. 156.

Hence, if h denote the height of this column, d the density of the mercury, and g the value of the acceleration due to gravity at the place of the experiment, the atmospheric pressure at that place is measured by hdg in units of force per unit of area.

Instead of expressing the atmospheric pressure in this way, in units of force per unit area, it is sometimes convenient to express it as the pressure due to a column of mercury of specified height, in centimetres or inches, the height so specified being the height of the column AB, which balances the atmospheric pressure, as in the foregoing experiment.

It must be carefully noted that the explanation given above turns on the fact that the pressure (**per unit of area**) in the

* This space, usually called the **Toricellian vacuum** (see Art. 53 in Part iv. on *Heat*), contains mercury vapour, and is, therefore, not a complete vacuum. At ordinary temperatures, however, the pressure exerted by the mercury vapour is negligible.

mercury inside the tube at the point A is equal, on the one hand, to the atmospheric pressure (**per unit of area**) at the surface of the mercury, and, on the other hand, to the pressure (**per unit of area**) due to the column of mercury AB.

It follows, therefore, since pressure per unit area only has to be considered, that the height of the column AB is quite independent of the extent of the surface of the mercury in the cistern, and also of the form and area of the cross-section of the tube.

The simple apparatus of Toricelli's experiment constitutes a simple form of **barometer** or instrument for measuring the atmospheric pressure.

In order to obtain an accurate measure of the pressure with this apparatus, or with any similar apparatus, it is, however, essential not only that the mercury should be clean and dry, but also that the mercury and the tube should be quite free from air. For this purpose it is necessary in filling the tube to boil the mercury in it until every trace of air is expelled. It is only by boiling the mercury in the tube that the film of air adhering to the walls of the tube can be completely removed.

A form of barometer, which is more convenient and portable than the tube and cistern form of Toricelli's experiment, is the **siphon barometer** shown in Fig. 157. The open end of the tube is bent upwards, as shown in the figure, so that the short, open limb, AB, takes the place of the cistern in Toricelli's apparatus. The height of the column of mercury, which measures the atmospheric pressure, is given in this form of barometer by the difference of the levels of the mercury at A and C in the open and closed limbs of the tube. This difference of level can be read off on a scale engraved on the tube, or on a wood or metal scale attached to the board on which the barometer tube is mounted.

The common dial form of the mercury barometer, the *weather-glass* in general use, is a form of siphon barometer. The mercury tube is set up in the long rectangular space at the back of the instrument. The indicator on the dial-plate on the front of the instrument is controlled by the arrangement shown in Fig. 158. The small pulley wheel W is mounted on the same axis as the indicator, and two small glass weights are slung by a thread over this wheel. One of these weights floats in the mercury in the open limb of the barometer tube, and the other serves as a counterpoise to the first. In this way, as

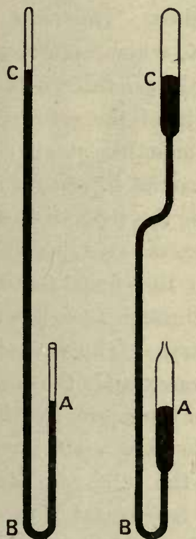


Fig. 157.

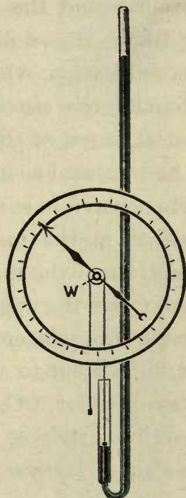


Fig. 158.

the mercury in the tube rises and falls, the wheel is made to revolve backwards and forwards by the thread, and the indicator, after once being correctly set, is made to indicate the height of the barometer on the dial-scale.

The most convenient form of mercury barometer for general use in accurate work is, however, the form due to Fortin, and known generally as **Fortin's barometer.**

This barometer is a tube and cistern barometer, similar to Toricelli's "simple barometer," but constructed so as to be readily portable, and provided with special appliances for the accurate measurement of the height of the barometric column of mercury.

The usual arrangement of the cistern and lower part of the tube of the instrument is shown in Fig. 159. The cistern is formed by the glass cylinder GGGG, the boxwood ring BB, which fits into the bottom of this cylinder, and the buckskin bag L, which closes the cistern below. This bag is tied securely round the boxwood ring XX, which screws into the ring BB, as shown in the figure; it is also provided with a small boxwood button, which rests on the tip of the screw S. The bottom of the cistern can thus be unmounted at any time by screwing it out of the ring BB which carries it, and its capacity can be decreased or increased by screwing the screw S up or down.

This cistern is enclosed in an outer metal case, which holds it together, in the manner indicated in the figure, and allows the surface of the mercury to be seen through the glass cylinder which forms the upper part of the cistern. The screw S works through the bottom of this outer case, and the barometer tube, drawn out to a point, as shown in the figure, fits through a boxwood collar, CC, in the cover. The tube is attached to this collar by a strip of buckskin tied to the tube and the collar; this strip of leather does not prevent free access to the air, but it prevents the escape of the mercury from the cistern.

In order to fix a definite, constant level, to which the surface of the mercury in the cistern can always be adjusted, a small ivory peg V is fitted through the cover of the outer case. The lower pointed tip of this peg marks the zero of the scale on which the height of the barometric column is read, and the surface of the mercury in the cistern must always be adjusted to touch this tip when a reading is taken. This is readily done by screwing the setting-screw S up or down until the tip of the peg and its image, seen by reflection at the surface of the mercury in the cistern, are seen to be coincident.

The tube of the instrument is surrounded, for protection, by an outer brass tube, indicated in dotted outline in Fig. 159. This outer tube extends to the top of the barometer tube, but in its upper part, at the level of the mercury column in the inner tube, two rectangular slits, about 20 cms. long, are

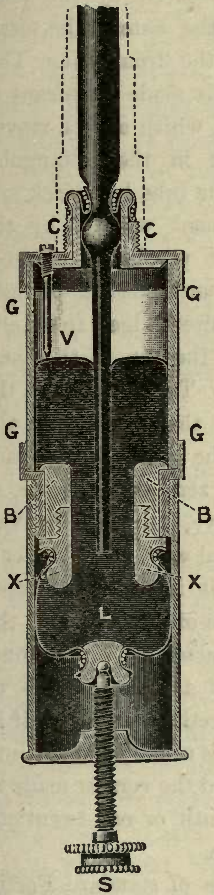


Fig. 159.

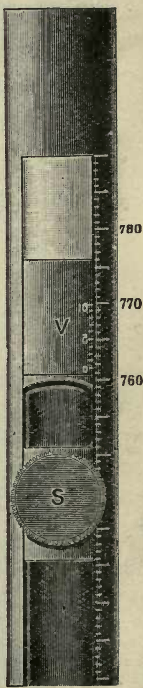


Fig. 160.

cut diametrically opposite each other and parallel to the length of the tube. This part of the tube is shown in Fig. 160. The

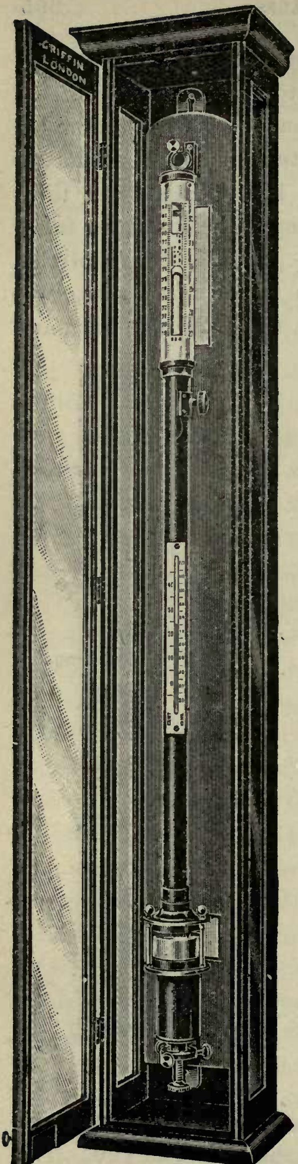


Fig. 161.

top of the mercury column in the inner tube is clearly seen through these slits, and the height of the column is measured by the scale engraved on the outer tube along the edge of the front slit. This measurement is made by means of the vernier **V**, which can be moved up and down in the rectangular space cut in the tube by a rack and pinion movement worked by the screw S. The vernier is adjusted by this screw until the line of sight through its lower edges, front and back, touches the top of the mercury column. The height of the column above the surface of the mercury in the cistern when adjusted to the zero of the scale, as explained above, can then be read off in the usual way. If the scale is in inches it is usually divided into twentieths of an inch, and the vernier constructed to read to one-twenty-fifth of a scale division, or to a five-hundreth of an inch; if in centimetres it is usually divided into millimetres, and the vernier made to read to one-tenth or one-twentieth of a millimetre.

A good form of standard Fortin barometer is shown in Fig. 161, enclosed in an outer glass case.

When the height of the barometric column, or *the height of the barometer*, as it is usually called, is accurately measured in this way by any good form of mercury barometer, it is subject to certain important *corrections* or *reductions* before it can be taken as the measure of the atmospheric pressure at the place of observation.

In the first place the effect of capillarity is evidently to depress the column of mercury in the tube, and so to reduce the height of the column below the height which is equivalent hydrostatically to the atmospheric pressure. This effect is, however, noticeable only in tubes of narrow bore, and may be neglected in tubes of wide bore, such as are used in standard barometers.

In tubes of narrower bore (less than 2 cms.) the necessary correction is best determined once for all by comparing the reading of the instrument with the corresponding reading on a standard barometer. This correction must in all cases be added to the observed height.*

The observed height is affected also by an error due to the variation of the length of the scale with change of temperature. The scale is usually constructed so that its divisions are correct at 0° C. Hence, if a denote the coefficient of linear expansion of the material of the scale, a division of the scale which is 1 unit of length at 0° C. is $(1 + at)$ units at t° C. That is, if the *observed* height of the barometer at t° C. is H units, its correct height is $H\ (1 + at)$ units. Then, if d_t denote the density of mercury at t° C. (the temperature of observation), and g the acceleration due to gravity at the place of observation, the atmospheric pressure at the place is $H\ (1 + at)\ d_t g$ units of force per unit of area.

It is not usual, however, to reduce the observed height to absolute units of force per unit of area in this way. It is found

* It should be noted that the error due to capillarity is eliminated in the siphon barometer if the tube is of the same bore at the upper and lower levels of the mercury.

more convenient to find the height of the column of mercury which would, at 0° C., at the sea level in latitude 45° C., exert the same pressure. Hence, if $\mathbf{H}$ denote the height of this column, d_0 the density of mercury at 0° C., and $\mathbf{g}$ the acceleration due to gravity at the sea level in latitude 45°, we must have

$$H\,(1 + at)\,d_t g = \mathbf{H} d_0 \mathbf{g},$$

where H is the observed height of the barometer at temperature t° C., and g the acceleration due to gravity at the place of observation as explained above.

This relation gives

$$\mathbf{H} = H\,(1 + at)\,.\frac{d_t}{d_0}\,.\frac{g}{\mathbf{g}}.$$

Now, $\dfrac{d_t}{d_0} = \dfrac{1}{1 + ct}$, where c is the coefficient of real cubical expansion of mercury.* We, therefore, have—

$$\mathbf{H} = H\,\frac{1 + at}{1 + ct}\,.\frac{g}{\mathbf{g}},$$

and since at and ct are small quantities, we may write—

$$\mathbf{H} = H\,[1 + (a - c)\,t]\,.\frac{g}{\mathbf{g}}.$$

Since c is, in general, greater than a, this is more conveniently written in the form—

$$\mathbf{H} = H\,[1 - (c - a)\,t]\,.\frac{g}{\mathbf{g}}.$$

In most barometers in general use the material of the scale is brass, and the value of a may be taken as ·00002. The value of c for mercury is ·00018, so that the value of $(c - a)$ is ·00016. The value of $\mathbf{g}$, the acceleration due to gravity at the sea level

* See Art. 21, Part iv., on *Heat.*

in latitude 45°, is 980·6 cms.-per-sec. per sec., so that we get as a final result

$$\mathbf{H} = H\,(1 - \cdot 00016\,t)\,\frac{g}{980 \cdot 6},$$

where H is the *observed height* at $t°$ C, at a place where the acceleration due to gravity is denoted by g, and **H** is the reduced or equivalent height at 0° C. at sea level in latitude 45° C.

Example.—The observed height of the barometer at 15° C. and at a place where the acceleration due to gravity is 981·2 cms.-per-sec. per sec., is 761·25 mms. Find the equivalent height reduced to 0° C. at the sea level in latitude 45°. (That is, find the *reduced height* of the barometer.)

Here, in the relation—

$$\mathbf{H} = H\,(1 - \cdot 00016\,t)\,\frac{g}{980 \cdot 6},$$

we have $H = 761 \cdot 25$, $t = 15$, and $g = 981 \cdot 2$.

We, therefore, have

$$\mathbf{H} = 761 \cdot 25\,(1 - \cdot 00016 \times 15)\,\frac{981 \cdot 2}{980 \cdot 6}.$$

That is,

$$\begin{aligned} \mathbf{H} &= 761 \cdot 25\,(1 - \cdot 0024)\,(1 \cdot 0006) \\ &= 761 \cdot 25\,(1 - \cdot 0018) \\ &= 759 \cdot 88. \end{aligned}$$

The *reduced height* of the barometer is thus found to be 759·88 mms.

If the height of the barometer is observed from day to day at any place it will be found to vary between comparatively wide limits. The atmospheric pressure at any place is, therefore, not of constant value, and it is necessary to specify a definite value as the *standard* or *normal* atmospheric pressure. This standard pressure is known as an *atmosphere* pressure, or a pressure of *one atmosphere*, and may be defined as the pressure due to a column of mercury 760 mm. high at 0° C. at the sea level in latitude 45° C. This standard pressure is equal to

$76 \times 13{\cdot}596 \times 980{\cdot}6$ dynes per sq. cm., or $1{\cdot}0132 \times 10^6$ dynes per sq. cm.

Examples.—The normal atmospheric pressure is sometimes defined (*a*) as the pressure due to a column of mercury at 0° C. 30 inches high at Greenwich ($g = 981{\cdot}17$), and (*b*) as the pressure due to a column of mercury at 0° C. 76 cms. high at Paris ($g = 980{\cdot}94$). Find the equivalent of each of these specified values in dynes per sq. cm.

If we take 1 inch = 2·54 cms., the standard pressure defined by (*a*) is equivalent to

$(30 \times 2{\cdot}54 \times 13{\cdot}596 \times 981{\cdot}17)$ dynes per sq. cm.,

or $(1{\cdot}0164 \times 10^6$ dynes per sq. cm.)

Similarly, the standard pressure defined by (*b*) is equivalent to

$(76 \times 13{\cdot}596 \times 980{\cdot}94)$ dynes per sq. cm.,

or $(1{\cdot}0136 \times 10^6$ dynes per sq. cm.)

It has been suggested that a convenient standard value for a pressure of one atmosphere would be a pressure of 10^6 dynes per sq. cm., but the suggestion has not yet been generally adopted.

Another form of barometer, which differs completely in principle and construction from the mercury barometer, is the **aneroid barometer.** The working parts of this barometer are shown in Fig. 162. It consists essentially of a small cylindrical box completely exhausted of air, so that the pressure inside the box is practically at zero value. The cover of the box is a thin, corrugated metal plate, B, which is forced inwards by the external atmospheric pressure to an extent which varies with the intensity of the pressure. The cover, therefore, moves in and out as the atmospheric pressure changes, and this movement of the cover is communicated directly to the spring D, which actuates the system of levers which controls the motion of the dial-index shown at F.

The aneroid barometer is specially useful under special conditions on account of its portability. It can be carried about from place to place like a watch or small clock, and the barometric height can be found at any place by simply taking the reading, indicated on the dial-scale, at that place.

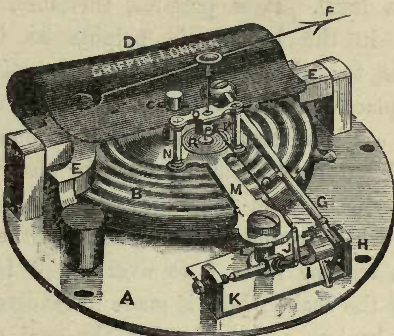


Fig. 162.

The scale on every reliable aneroid is obtained, in the first instance, by comparing the readings of the instrument with

Fig. 163.

those of a standard mercury barometer. A convenient form of aneroid, about the size of a large watch, is shown in Fig. 163.

Direct observations with the barometer show that the atmospheric pressure decreases, as explained above, with height above the sea level. It is possible, therefore, by observing the difference in the barometric heights at two places at different levels to calculate their difference of level or the height of one place above the other. The height of a mountain-top above the sea level may, for example, be found in this way.

The aneroid barometer is convenient for observations of this kind, and it will be seen that the small aneroid, shown in Fig. 163, is provided, round the outer circle of the scale, with an altitude scale, reading from 0 to over 8,000 feet. In using the instrument the scale, which is made to revolve, is set with a key, or by a keyless action, so that the zero of the scale is opposite the height of the barometer at 0° C. at the sea level at the time and place of observation. The reading of the scale at any point then gives directly the height of the point above the sea level.

123. **Boyle's Law.**—It has already been stated in Art. 92 that the pressure which a given mass of any gas exerts at a given temperature depends upon the volume which it occupies.

The relation between the pressure and the volume of a given mass of a gas at *constant temperature* is found to conform generally to a certain definite law.

It is found that the volume which a fixed quantity of a gas occupies at constant temperature is inversely proportional to its pressure. That is, if a fixed quantity of gas at any constant temperature occupies a volume V_1 under a pressure P_1, and a volume V_2 under a pressure P_2, then the relation between pressure and volume at constant temperature is such that

$$\frac{V_1}{V_2} = \frac{P_2}{P_1}.$$

This law was discovered by Boyle in England in 1662, and also, quite independently, in 1676, by Mariotte in France.

The law is, therefore, known in England as **Boyle's Law**, or Boyle and Mariotte's Law, and in France as **Mariotte's Law.**

It will be seen from the statement of the law given above that if a fixed quantity of gas at constant temperature has a volume V_1 under a pressure P_1, and a volume V_2 under a pressure P_2, then

$$\frac{V_1}{V_2} = \frac{P_2}{P_1},$$

or

$$P_1V_1 = P_2V_2.$$

It follows from this if a fixed quantity of gas at constant temperature has volumes denoted by V_1, V_2, V_3, . . . V_n, under pressures denoted respectively by P_1, P_2, P_3, . . . P_n, then we must have

$$P_1V_1 = P_2V_2 = P_3V_3 = \ldots = P_nV_n.$$

That is, if V denote generally the volume of the gas under pressure P, then *for all corresponding vulues of V and P*, we must have

$$\mathbf{PV} = k,$$

where k is a **constant.**

This result may be deduced directly from the statement of the law as given above. According to this statement we have

$$V \text{ varies as } \frac{1}{P}.$$

That is, $V = k \cdot \frac{1}{P}$, where k is a constant, and, therefore, $PV = k$.

This form of the relation between P and V is conveniently expressed graphically; for if a curve is plotted, as in Fig. 164, so that the abscissæ represent the volumes, and the ordinates the corresponding pressures of a given mass of gas at a constant temperature, the form of the curve obtained is the characteristic *rectangular hyperbola* shown in the figure.

It will be seen that Boyle's law may be stated in terms of the pressure and density of the gas. For, if a given mass of a gas has a volume V_1 and density D_1 under a pressure P_1, and a volume V_2 and density D_2 under a pressure P_2, then since

$$\frac{D_2}{D_1} = \frac{V_1}{V_2}, \text{ and } \frac{V_1}{V_2} = \frac{P_2}{P_1},$$

we must have

$$\frac{D_1}{D_2} = \frac{P_1}{P_2}.$$

That is, the density of a gas at constant temperature is directly proportional to its pressure.

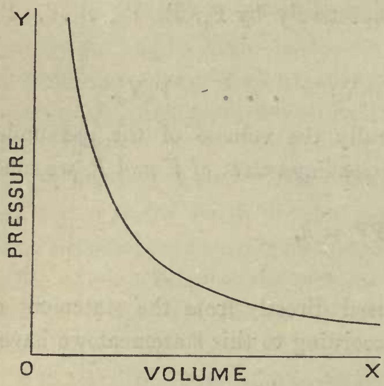


Fig. 164.

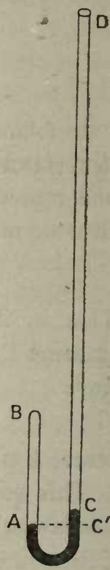


Fig. 165.

Boyle's law may be studied experimentally, in the case of air, by the following simple method. A large U-shaped tube with a long open limb, CD, and a shorter closed limb, AB, as shown in Fig. 165, has a quantity of mercury poured into the bend AC, so as to separate a quantity of air in the closed limb AB from the external air. The volume of this air can be measured either by means of suitable graduations on the tube or simply by measuring the length of the tube occupied by the air. This latter method assumes, however, that the bore of the tube is uniform, and the unit in which the volume is measured is evidently the capacity of the bore per unit length.

The pressure to which the air in AB is subject is evidently given by the atmospheric pressure exerted on the surface of the mercury at C, *plus* or *minus* the pressure due to the difference of the mercury levels at A and C, according as the level at C is higher or lower than the level at A. Thus, in Fig. 165, the pressure of the air in the closed limb is the atmospheric pressure at C, *plus* the pressure due to the short column CC′ of mercury. It will be seen that this pressure may be increased to an extent which is limited only by the height of the limb CD by pouring mercury into this limb.

It is possible, therefore, by increasing the pressure in this way, step by step, and observing the corresponding values of the pressure and volume at each step, to determine experimentally the relation between the pressure and volume of air in the tube AB at constant temperature.

The tube shown in Fig. 165 is similar to that used by Boyle in the experiments by which he established the law. It usually takes the form shown in Fig. 166, and is generally known as **Boyle's tube.** Both limbs of the tube are usually graduated in the same way in length divisions (preferably mms.) from a zero at the same horizontal level on both tubes.

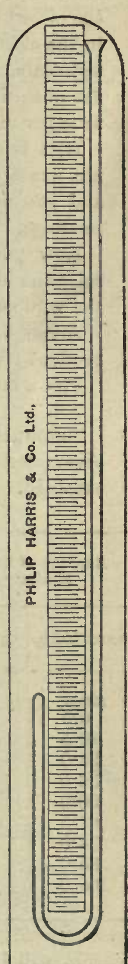


Fig. 166.

Experiment 33.—Take a Boyle's tube, similar to that shown in Fig. 166, pour a quantity of mercury into the bend, and adjust the level of the mercury in the closed limb to the zero level of the scales on the tube, by tilting the tube and allowing air to enter or escape from the closed limb as may be necessary.

Read the height of the barometer, and also the height of the level of the mercury in the open limb above the zero level. The sum of these two readings (in mms. or inches) is the pressure of the air in the closed limb in mms. or inches of mercury. The volume of the air in the closed limb may be recorded in terms of the linear divisions of the scale, but it must be remembered, as explained above, that the unit of volume thereby adopted is the capacity of the bore of the tube per division.

If it is found that the level of the mercury in the closed limb is not exactly adjusted to the zero level, the actual level must be read, and the necessary corrections applied to the pressure and volume readings.

Now pour mercury into the open limb, step by step, so as to raise the level above 50 mms. at each step, and read the pressure and the volume of the air in the closed limb at each step.

Arrange the readings taken in tabular form, and verify that the product PV for the air in the closed limb is practically constant.

Plot a curve showing the relation between P and V, as in Fig. 164.

Example.—In an experiment, similar to that described above, the observations and results set out in the table given below were obtained.

Reading of Level of Mercury in		Height of Barometer.	Pressure of Air in Closed Limb. P.	Volume of Air in Closed Limb (Scale extends from 0 – 200 mm.). V.	Value of Product. PV.
Closed Limb.	Open Limb.				
mms.	mms.	mms.	mms. of mercury.	Scale units.	
0·5	2·5	756	758	199·5	151,200
10·5	52·5		798	189·5	151,200
25·0	133·0		864	175·0	151,200
36·5	205·5		925	163·5	151,200
50·0	302·0		1,008	150·0	151,200
61·5	397·5		1,092	138·5	151,200
74·0	518·0		1,200	126·0	151,200
85·0	644·0		1,315	115·0	151,200
100·0	856·0		1,512	100·0	151,200

It will be seen from these results that the value of PV is practically constant, and it may be noted from the first and last readings that when the pressure of the air in the closed limb is *doubled*, the volume is *halved*.

The constant in the last column is given only to the fourth significant figure. It will generally be found, in a rough experiment of this kind, that the agreement between the different values does not go beyond the second or third significant figure. The agreement is, however, sufficient to indicate the general truth of the law.

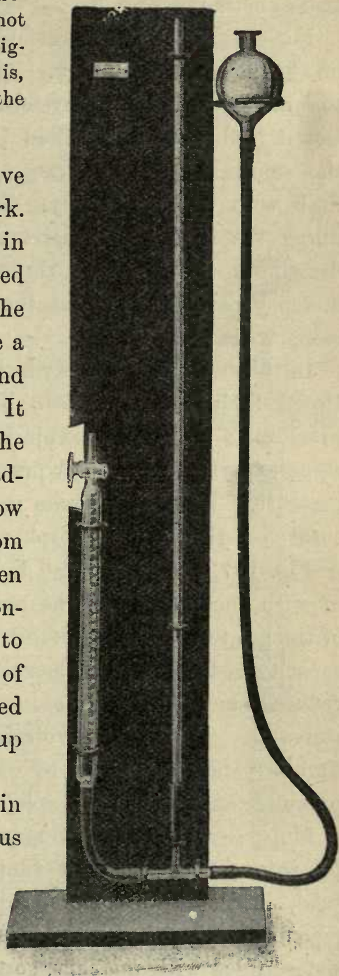


Fig. 167.

The type of tube described above is not suitable for accurate work. A better form of tube is shown in Fig. 167. In this form the closed limb is fitted with a stopcock at the closed end, and is graduated, like a burette, in cubic centimetres and tenths of a cubic centimetre. It is also arranged, as shown in the figure, so that mercury can be admitted into the tube from below instead of being poured in from above at the upper-end of the open limb. The mercury supply is contained in a reservoir connected to the bend of the tube by a length of stout rubber-tubing, and is carried by a metal holder which slides up and down the stand as required.

The pressure to which the gas in the closed limb is subject can thus be varied by raising or lowering the mercury reservoir. If the stopcock on the closed limb is opened at the beginning of any set of observations the initial volume of the air in the closed limb can be adjusted to any desired value, and the initial pressure will be the existing atmospheric pressure.

For exact work it is necessary that the air or gas experimented on should be perfectly dry. The inner surface of the tube and the mercury should, therefore, be thoroughly dried, and the closed limb should be filled by drawing in the air or gas at the stopcock through a range of drying tubes connecting the limb with the gas supply. This may easily be done by first filling the limb with mercury up to the stopcock, and then drawing in the air or gas, through drying tubes, by lowering the mercury reservoir, so that the dry gas enters the limb as the mercury leaves it.

In the experiments described above, the volume of gas in the closed limb is supposed to be varied only by *increasing* the pressure to which it is subjected. That is, the law is tested by these experiments only for pressures *greater* than the atmospheric pressure. The experiments may, however, be extended to pressures less than one atmosphere by arranging the tube, shown in Fig. 167, with the closed limb on a level with the *top* instead of with the bottom of the open limb. If the initial volume of the gas in the closed limb is then adjusted to about half the capacity of the limb at atmospheric pressure, the pressure can be reduced, and the volume increased by lowering the mercury reservoir. As the reservoir is lowered the gas in the closed limb expands, and, step by step, readings of the volume and pressure can be taken as explained above.

More accurate results may, however, be obtained at low pressures by the method adopted by Regnault. A graduated tube, similar to a barometer tube, is nearly filled with clean dry mercury, and then inverted, as shown in Fig. 168, in a tubular cistern filled with mercury, so that the small quantity of air left in the tube at filling rises to the upper part of the tube, and occupies the space AB above the mercury in the tube. The volume of this small quantity of air in AB can be read off on the graduations of the tube, and its pressure is evidently the atmospheric pressure on the surface of the mercury in the cistern, *minus* the pressure due to the column BC of mercury. The volume and pressure of the air, which can thus be determined for any position of the tube in the cistern, can evidently be varied by raising or lowering the tube in the cistern. As the tube is lowered, for example, the volume decreases, and the pressure increases and becomes equal to the atmospheric pressure when the volume of the air in AB reaches its initial value at the time of filling the tube.

The volume and pressure of the air can, therefore, be observed for a series of successive positions of the tube, and the truth of Boyle's law, for pressures less than one atmosphere, can be tested and established.

Fig. 168.

Experiment 34.—Take a clean dry barometer tube and pour clean dry mercury into it until the level of the mercury rises to within a few inches of the mouth of the tube. Measure carefully the length of part unoccupied by mercury: this gives the volume of the air which fills the space, and, as the air is at the atmospheric pressure, its volume and pressure are both known.

Now place the thumb, or a flat rubber-pad, over the mouth of the tube, and invert it in an ordinary cistern of mercury, so that the air rises to the upper part of the tube. Measure the length now occupied by the air, and determine its pressure by subtracting the height of the column of mercury standing in the tube from the barometric height.

Verify that the product PV for the quantity of air in the tube is the same for the values of P and V observed before inverting the tube, as for the values observed after inversion.

Repeat the experiment for a number of different quantities of air.

Note that the value of PV is constant for any one quantity of air, but the value of the constant is different for the different quantities.

After Boyle's law was first established by Boyle and Mariotte it became the subject of much careful experimental research with the object of discovering whether the law was strictly exact and applicable to all gases.

The first result which was conclusively established was the fact that different gases, such as air, oxygen, hydrogen, nitrogen, carbon dioxide, and sulphur dioxide, when subjected to the *same* increase of pressure, were compressed to *different* extents. This showed at once that the law could not apply exactly to every gas even if it applied exactly to some one gas.

Dulong and Arago, in 1826, and Regnault, in 1847, carried out very careful and laborious investigations of the truth of the law by the simple method described above, but with much more accurate and elaborate apparatus. The open limb of the tube was extended in Regnault's apparatus to over thirty metres in height, so as to extend the range of the experiments to high pressures, and the mercury was pumped into the tube from below instead of being poured in from above.

An important improvement introduced by Regnault in the method of the experiment enabled him to avoid the comparatively large percentage error which attends the measurement of the volume of the air under high pressure. As the pressure is increased the volume of the air decreases, and at high pressures the volume becomes so small that the inevitable error made in observing it may become a very large fraction of the observed value. Regnault avoided this error by taking the same initial volume of gas at each pressure, and reducing it by increase of pressure to the same final volume (about half the initial volume) in each case. Every time the volume of the gas in the closed limb was reduced to this final volume, more gas was pumped into the limb until the initial volume was again reached, and this volume was then, in turn, reduced again to the final volume by applying the necessary increase of pressure.

Neither Dulong and Arago, nor Regnault were able to deduce any *general result* from their investigations. It was shown, however, conclusively that different gases deviate to different extents from exact conformity to the law. It was found, for example, that all gases, except hydrogen, are slightly more compressible under increase of pressure than they would be if they obeyed the law exactly, and that this deviation was more marked at ordinary temperatures in the case of gases, such as carbon dioxide and sulphur dioxide, which are not far removed from their temperatures of liquefaction, than in gases such as oxygen and nitrogen. Hydrogen, on the other hand, was found to be less compressible under increase of pressure than is required by the law.

These results, although well established by experiment, are evidently of an *empirical*, rather than a *general* nature, and it was not until 1870 that the general nature of the deviation of all gases from strict obedience to Boyle's law was formulated. In this year **Amagat** published the result of a very complete research on the truth of Boyle's law, and the nature of the deviations from it in the cases of a number of different gases. It was found that in *all* gases the value of PV, instead of being a constant for all values of P, first decreased to a minimum and then increased as the value of P increased. The value of P, for which PV reached its minimum value, and the rates of decrease and increase before and after reaching the minimum were found to vary within somewhat wide limits for different gases, but the general nature of the variation of PV with P was the same for all gases.

These results are most effectively exhibited by means of curves showing the relation between PV and P for different gases. The curves for hydrogen, nitrogen, and carbon dioxide at several different temperatures are given in Figs. 169, 170, and 171.

It will be seen that the minimum value of PV occurs at very different values of P for different gases, and that it is a much more marked and characteristic feature of the curve for an easily liquefiable gas, such as carbon dioxide, than for the more "permanent" gases, such as hydrogen and nitrogen.

It will be seen, too, from the carbon dioxide curves in Fig. 171,

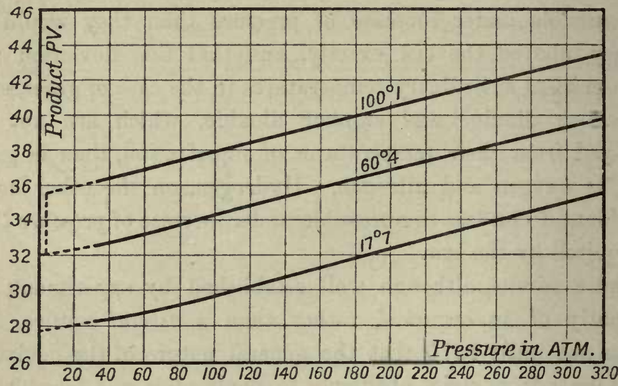


Fig. 169.—Hydrogen.

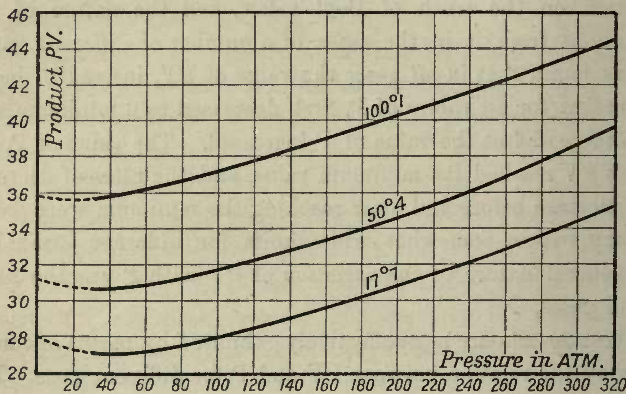


Fig. 170.—Nitrogen.

that the deviation from Boyle's law, and the minimum value of PV, become more and more marked as the temperature approaches the temperature of liquefaction.

In the case of hydrogen it is to be noted that the minimum value of PV corresponds to a very low value of P, so that the value of PV for this gas appears at all ordinary pressures to increase with P.

If we consider the variation of PV with P for the gases

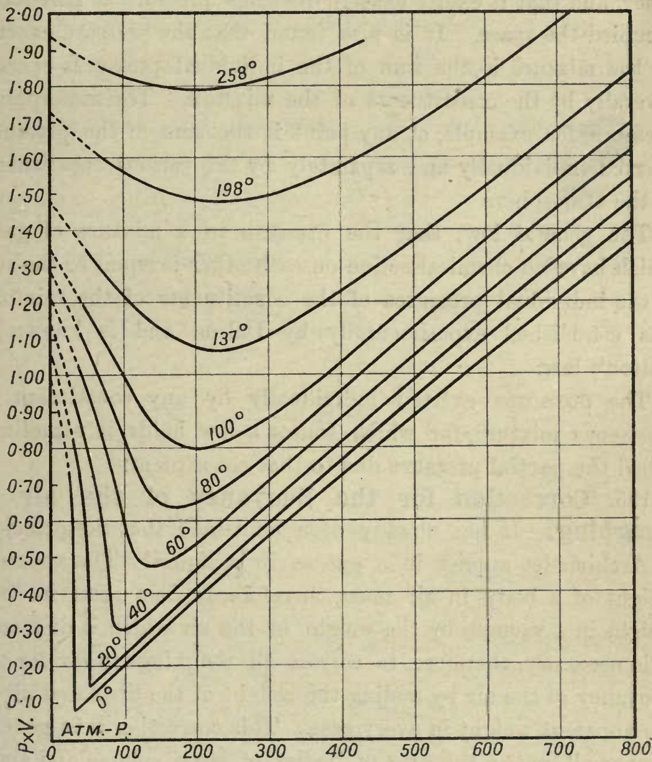


Fig. 171.—Carbon dioxide.

for which curves are given in the figure, it will be seen that for a limited range extending, say, from one to ten atmospheres, the value of PV *decreases* with P for all gases, except hydrogen, and *increases* for hydrogen. These are obviously the results given by Regnault, which are thus seen to be included in Amagat's more complete and general result.

124. **Dalton's Law.**—When a mixture of gases, which have no chemical action on each other, occupy the same space, it is found that each gas is distributed uniformly throughout the space, and that it exerts exactly the same pressure as if it alone occupied the space. It is also found that the pressure exerted by the mixture is the sum of the individual pressures exerted severally by the constituents of the mixture. The atmospheric pressure, for example, at any point is the sum of the pressures exerted individually and separately by the various constituents of the atmosphere.

The general law, that the pressure in a mixture of gases which have no chemical action on each other is equal to the sum of the individual pressures of the constituents of the mixture, was established experimentally by Dalton and is known as **Dalton's law.**

The pressures exerted individually by any constituent of a gaseous mixture, for which Dalton's law holds, is sometimes called the **partial pressure** due to that constituent.

125. **Correction for the Buoyancy of the Air in Weighing.**—It has already been explained that the principle of Archimedes applies in a gas as in a liquid. The apparent weight of a body in air must, therefore, be less than its true weight in a vacuum by the weight of the air which it displaces. It is necessary, therefore, to correct all weighings in air for the buoyancy of the air by adding the weight of the displaced air to the apparent weight in every case. This correction is in general very small, but in the case of bodies of large volume and small mass the weight of the displaced air may be comparable with the weight of the body, and the correction becomes an important one.

The value of the correction in any given case is usually found in the following way. When a body is weighed in air in the usual way the balance obtained evidently indicates that *the apparent weight of the body in air is equal to the apparent weight of the weights in air.* Now, the nominal value, or the value marked on any accurate weight, is usually its true weight *in vacuo*, so that, if W denote the observed weight of the body, as indicated by the nominal value of the weights in the scale pan, and d denote the density of the material of the weights, the apparent weight of the weights in air is $\left(W - \frac{W}{d} . \delta\right)$, where δ denotes the density of the air. Similarly, if W' denote the true weight of the body *in vacuo*, and d' the density of its material, its apparent weight in air is $\left(W' - \frac{W'}{d'}\right) \delta$.

We, must, therefore, have

$$W - \frac{W}{d} . \delta = W' - \frac{W'}{d'} \delta,$$

or $$W\left(1 - \frac{\delta}{d}\right) = W'\left(1 - \frac{\delta}{d'}\right),$$

and $$W' = W \frac{1 - \frac{\delta}{d}}{1 - \frac{\delta}{d'}}.$$

The true weight W' can thus be calculated from the observed weight W if d, d' and δ are known. The value of δ varies from day to day with the temperature, pressure, and humidity of the air, so that for accurate work its value in any particular case would have to be calculated from observations of these atmospheric data at the time of weighing. For ordinary purposes, however, the value of δ may be taken as 0·0012 gramme per c.c., and, as the weights in general use are made of brass, the value of d' is about 8·4 grammes per c.c. If we substitute these values in the relation given above, we get

$$W' = W \frac{(1 - 0{\cdot}00014)}{\left(1 - \frac{0{\cdot}0012}{d'}\right)},$$

where W and W′ are expressed in grammes.

In the calculation given above it is assumed that the body weighed is of uniform density throughout its mass. When this is not the case—as, for example, in the case of a large glass globe completely or partially exhausted of air—the volume of the displaced air must be found by determining the external volume of the body. This may be evidently done by finding directly the weight of water which the body displaces, or by finding its apparent loss of weight in water.

The relation given above will then become—

$$W\left(1 - \frac{\delta}{d}\right) = W' - Vd',$$

or

$$W' = W\left(1 - \frac{\delta}{d}\right) + Vd',$$

where V denotes the volume of the displaced air.

126. **Determination of the Density of Gases.**—The density of a gas is usually determined by finding the mass of unit volume of the gas directly. The mass of a known volume of the gas is found by finding the weight of the gas which fills a glass bulb of known capacity at a known temperature and pressure.

A spherical glass bulb or globe provided with a capillary glass stopcock is completely exhausted of air, and weighed; it is then filled at known temperature and pressure with the gas whose density is to be determined, and again weighed. The increase in weight gives the mass of the gas which fills the globe at the temperature and pressure of filling. In order to eliminate the correction due to the buoyancy of the air in these weighings the globe is usually counterpoised by a similar globe of exactly equal external volume; the air displaced by each globe is thus of the same weight, and the buoyancy correction need not be considered.

The capacity of the bulb is found by determining, in exactly the same way, the mass of water which fills it at a known temperature.

From these weighings the absolute density of the specific gravity of the gas may be determined.

In some cases the density of the gas is found relative to some standard gas, such as oxygen, instead of its density relative to water. In order to do this it is necessary to find the mass of the standard gas which fills the density globe at a known temperature and pressure.

The capacity of the globes used for the determination of the density of a gas need not be more than 50 c.c., except in the case of a gas of very low density, such as helium or hydrogen, when a capacity of nearly 200 c.c. is desirable.

In accurate determinations of the density of gases it has been found necessary to make a correction for the decrease in the volume of the density globe when exhausted of air. The decrease due to the compressing effect of the external pressure affects the accuracy of the result obtained to an extent which is far from negligible.

The absolute densities of a few of the common gases at normal temperature and pressure are given below in grammes per cubic centimetre:—

Air,	1·2932 grammes per c.c.	Nitrogen,	1·2571 grammes per c.c.
Oxygen,	1·4293 ,, ,,	Hydrogen,	0·0899 ,, ,,

127. **Diffusion.**—The process of diffusion in gases is of a very similar character to that of diffusion in liquids. If a vessel contains two or more gases, none of which is uniformly distributed throughout the available space in the vessel, mole-

cules of each gas will pass from points of higher to points of lower density or pressure for that particular gas, until it is in hydrostatic equilibrium at all points throughout the space occupied by the gases. This process of mixing of gases by the migration of the molecules from points of higher to points of lower density for each constituent is called **diffusion.** The process is much more rapid in gases than in liquids, and takes place in every direction irrespective of the action of gravity and the relative density of the gases.

The uniform distribution * of the constituents of the atmosphere throughout the space which it occupies is due to diffusion, and the fact that the composition of the atmosphere is the same at all points and at all levels, shows that diffusion takes place equally in all directions for all its constituents.

The quantitative treatment of diffusion in gases follows the same line as that adopted for diffusion in liquids. Imagine each gas in a mixture of gases to be arranged in parallel plane layers of uniform density, and let δ denote the difference in density between two adjacent layers a very small distance x apart, then $\frac{\delta}{x}$ is the gradient of density between these layers, and the quantity of gas which diffuses across unit area in unit time from one layer to the other, is proportional to this gradient, and is, therefore, denoted by $k\frac{\delta}{x}$, where k is a constant. This constant k for a given gas in a given mixture is the *coefficient of diffusion* of the gas under the given conditions. Experiments for the determination of k for a given gas have been made only for mixtures of that gas with one other gas, and the coefficient is then known as the coefficient of inter-diffusivity of the two

* Uniform distribution must here be understood to mean that the ratio of the *partial pressure* due to any constituent at any point, to the *atmospheric pressure* at that point, is the same for all points in the atmosphere.

gases. In one method of experiment for the determination of this coefficient, a long cylinder is divided into two halves by a central partition, and one half is filled with one gas and the other half with the other gas. The partition is then removed without disturbing the gases, and mixing by diffusion allowed to go on for a known time. The partition is then replaced, and the quantity of each gas which has diffused from one half into the other is determined by analysis. From results obtained in this way the coefficient k can be calculated.

It can be shown that if the coefficient of diffusion for a gas is defined as above, and if at any point in the mixture π denote the difference of partial pressure corresponding to the difference of density δ, then the rate at which the gas diffuses at that point is directly proportional to π, and inversely proportional to the square root of the density of the gas.

The passage of a gas through a porous partition is usually considered under the head of diffusion. It is, however, important to understand that a gas may pass through a porous partition by three different processes. If the holes or pores in the partition are not very fine, and are large compared with the thickness of the partition (as would be the case, for example, in a partition of metal foil perforated by very small holes made by a needle point), the gas passes through the partition by the process of **effusion**. This process is the same as that by which a liquid flows out through a hole made in the wall of the vessel containing it. The theory of the process cannot here be considered, but it may be stated that the rate of flow of the gas through the partition is proportional to the difference in the pressure of the gas on the two sides of the partition, and inversely proportional to the density of the gas.

If the holes or pores in the partition are not very fine, but are small compared with the thickness of the partition, the gas passes through by the process of *transpiration*. This process is the same as that by which a fluid passes through a capillary tube, and the rate at which it takes place depends, as explained in Art. 117, on the viscosity of the fluid.

If, however, the pores of the partition are fine enough to be of molecular dimensions, the gas passes through by a process which is practically identical with diffusion as considered above. The rate at which a gas passes through a partition of this kind will, therefore, be proportional to the difference of the partial pressures due to the gas at the two faces of the partition, and inversely proportional to the square root of the density of the gas. Thus, if two gases diffuse through a partition under exactly the same conditions of pressure and temperature, the rate at which they severally pass through the partition are inversely proportional to the square roots of their densities. For example, if two gases, *at the same pressure and temperature*, are separated by a porous partition through which each gas can pass by diffusion, the rates at which the gases *begin* to diffuse through the partition, are inversely proportional to the square roots of their initial densities. As diffusion proceeds, however, the difference between the partial pressures at the opposite faces of the partition ceases to be the same for each gas, and the ratio of the rates of diffusion is determined by the more general rule given above.

The law that the rates of diffusion of different gases under the same conditions of pressure, gradient, and temperature are inversely proportional to the densities of the gases, is *Graham's law of diffusion*, and was established by Graham experimentally from observation of the rates of diffusion of gases through partitions of porous materials such as meerschaum, compressed graphite, and plaster of paris.

When a mixture of gases passes through a partition by *effusion* or *transpiration*, no separation or partial separation of the constituents of the mixture takes place; the mixture passes through as a mixture, and its composition is practically unchanged by its passage through the partition. When, however, a mixture of gases is allowed to *diffuse* through a partition, the different constituents in general diffuse at different rates, and a partial separation of the constituents may thus be produced. The gas which diffuses through the partition will obviously be richer than the initial mixture in those constituents which diffuse most rapidly, while the gas which has not passed through at any stage in the process will be richer in those constituents which diffuse most slowly.

This process of partial separation of the constituents of a gaseous mixture by diffusion through a porous partition is called **atmolysis**. In practice the mixture is usually passed through a long stem of a clay tobacco pipe enclosed in a tube from which the air is exhausted. A slow current of the mixture is passed through the stem, and diffusion of the several constituents takes place through the wall of the stem into the vacuum in the surrounding tube. The composition of the mixture may thus be appreciably changed by its passage through the pipe stem, for the mixture which emerges from the stem will be richer than the initial mixture in those constituents which diffuse most slowly through the stem wall. A notable instance of the application of this method is found in its use by Sir William Ramsay and Lord Rayleigh in the partial separation of nitrogen and argon. A mixture of these gases, containing only a small percentage of argon, was passed through a considerable length of pipe stem as explained above. The nitrogen being greatly in excess, its partial pressure in the mixture was much greater than that of the argon, and its density being also less than the density of argon, it diffused through the wall of the stem much more rapidly, so that the gas collected at the other end of the stem, after passing through its whole length, was much richer in argon than the initial mixture.

128. **Mechanical Air Pumps.**—An air pump is an instrument constructed for the purpose of pumping air or any similar

gas out of a closed vessel, and so producing a more or less complete *vacuum* in the vessel. Pumps of this kind are of two distinctly different types; one type being known as the *mechanical pump*, and the other as the *mercury pump*.

The general construction and action of the mechanical pump are indicated in Fig. 172. It consists of a cylindrical metal barrel, AB, in which a lightly fitting piston, C, can be worked up and down by the handle D at the upper end of the piston

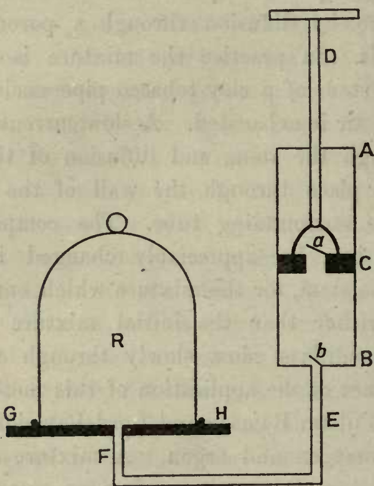


Fig. 172.

rod. This barrel communicates through the tube EF with the vessel to be exhausted of air, or, as shown in the figure, with a bell-jar receiver, R, which fits as an airtight cover on the flat circular plate GH.

The piston C is fitted at *a* with a valve, which opens upwards only,* and a similar valve is fitted also at *b*, at the mouth of the tube EF, which passes from the barrel to the receiver.

* A small hole covered with a stretched strip of thin sheet rubber forms a simple value of this kind.

It will be seen that when the piston is forced downwards from the top to the bottom of the cylinder, it tends to compress the air in the barrel below it. This at once closes the valve at b, and when the pressure in the cylinder becomes greater than the atmospheric pressure, the valve at a opens, and the air is forced out through it into the upper part of the cylinder and thence to the outer air. In this way practically the whole of the air in the cylinder is driven out through the valve a by the *downstroke* of the piston. Then, when the piston is raised for the *upstroke*, the valve at a at once closes, and the air in the receiver expands into the cylinder as the piston is raised, so that at the end of the upstroke the air which filled the receiver only at the beginning of the stroke, now fills the receiver and cylinder. At the next downstroke the whole of the air in the cylinder will again be expelled, and at the following downstroke the air left in the receiver will again expand so as to fill both receiver and barrel. This process of *exhaustion* goes on, stroke after stroke, until a fairly low vacuum is produced in the receiver.

The theory of this process is comparatively simple. Let V denote the volume of the receiver and tube up to the valve at b, and v the volume of the cylinder from the valve at b to the valve at a when the piston is at the top of its upstroke. Then, since at each double stroke, consisting of a downstroke, followed by an upstroke, the air occupying a volume V in the receiver expands, and occupies a volume $V + v$ in the receiver and cylinder, it follows that if D and d denote respectively the density of the air at the beginning and at the end of the stroke, we must have

$$VD = (V + v)\,d,$$

or

$$d = \frac{V}{V + v} \, . \, D.$$

That is, the density d at the end of the stroke is equal to $\frac{V}{V+v}$ times the density at the beginning of the stroke.

Hence, if D denote the density of the air in the receiver at the beginning of the process of exhaustion, the density of the air at the end of the *first* double stroke will be given by

$$d_1 = \frac{V}{V+v} \cdot D.$$

Similarly, the density at the end of the *second* double stroke will be given by

$$d_2 = \frac{V}{V+v} d_1 = \left(\frac{V}{V+v}\right)^2 D,$$

and it will be seen by continuing this line of argument, that the density at the end of the n^{th} double stroke is given by

$$d_n = \left(\frac{V}{V+v}\right)^n D.$$

This result shows that the value of d^n can never become zero (indicating a perfect vacuum) no matter how many strokes are made, but it is obvious that if the pump is mechanically perfect the value of d_n may be made negligibly small by a comparatively small number of strokes.

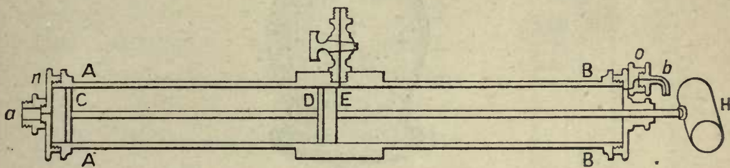
Pumps of the pattern indicated by Fig. 172 are, however, very far from being mechanically perfect, and are rapidly going out of use in modern practice. There is always a certain amount of leakage in action, and it is impossible to make the bottom of the piston fit so closely to the bottom of the cylinder as to drive out *all* the air in the cylinder at the end of the downstroke.* Another difficulty arises out of the fact that when the pressure of the air in the receiver becomes very low it is unable to raise the valve at b during the upstroke of the

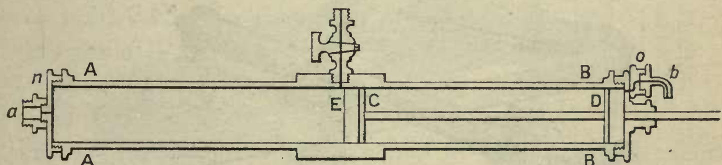
* That is, there is always a small *clearance* or space between the bottom surfaces of the piston and cylinder.

piston. This difficulty can be remedied by arranging for the valve to be opened and closed mechanically by the action of the piston, but this complicates the construction of the instrument, and adds to the risk of leakage.

One of the best of the mechanical pumps of the piston and valve pattern is the **Tate pump,** shown diagrammatically in section in Figs. 173 and 174.

The barrel or cylinder AB is fitted with a double piston, CD, of which the length is rather less than half the length of the cylinder. This cylinder communicates at its middle point, E, with the receiver. Two valves, both opening outwards, are





Figs. 173 and 174.

provided at *a* and *b* at the ends of the cylinder, and the piston is arranged to be worked backwards and forwards in the cylinder by means of the handle shown at H.

Imagine the piston to be drawn out the full length of its stroke from the extreme position at the end A, as shown in Fig. 173, to the corresponding extreme position, indicated at the end B, as shown in Fig. 174. It will be seen that as the piston is drawn out from A towards B, the communication between the receiver and the half DB of the cylinder is cut off, and the air in this half is forced out through the valve at *b*. At the same time the valve at a closes, and a vacuum is formed in the space CA, between the piston and the end of the cylinder, until the end C of the piston passes the point D, and the air in the receiver is free to expand into this space. At the return stroke of the piston the process just described is reversed; the air in the half DA is driven out at the valve a, and a vacuum is formed in the space DB until the piston is pushed home, and the receiver is again put in communication with the half DB of the cylinder.

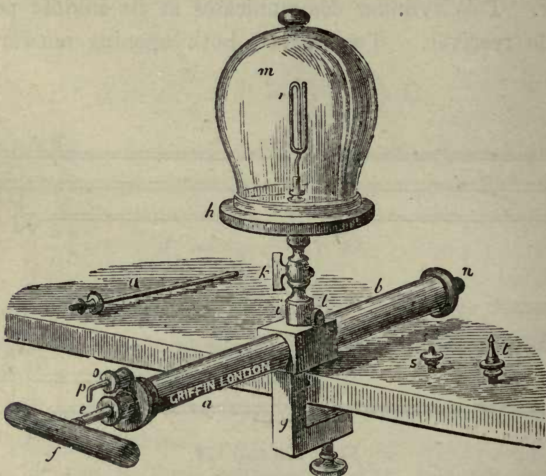


Fig. 175.

It thus appears that every stroke of the piston is a double stroke, for each half of the cylinder acts as a separate cylinder. In the case of this pump the volume of the receiver, denoted by V in the general theory given above, is evidently the volume up to the point E, and the volume of the cylinder, denoted by v, is the volume of the space between the piston and either end of the cylinder when the piston is in its extreme position at the other end.

A general view of a good form of this pump is shown in Fig. 175.

A more satisfactory form of mechanical pump which has come into general use in recent years is the **Fleuss pump.** This pump is practically free from the defects due to valve action and piston clearance, and is found to give a much better vacuum than the older forms described above. The general construction of this pump is shown diagrammatically in Fig. 176.

The cylinder AB is fitted with a piston, C, and a fixed partition, D, through which the piston-rod works. This partition is a little below the cover of the cylinder, and is provided with a value at *a*, which opens outwards. The cylinder contains a quantity of oil sufficient to fill it up nearly to the level of the side tube E when the piston is at the bottom. The tube E communicates through the tube G with the receiver or vessel to be exhausted, and the tube F is provided to allow the oil to flow round from the under to the upper side of the piston at the end of each downstroke. The safety bulb at H is intended to prevent the oil finding its way into the tube G if the piston is forced down too suddenly at the end of its stroke.

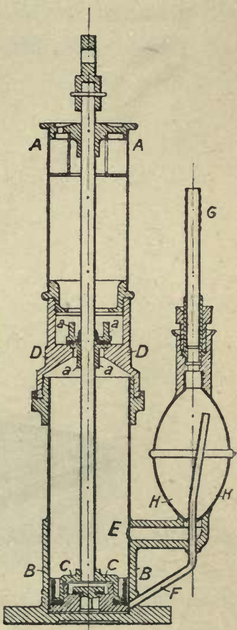


Fig. 176.

If we imagine the piston to be raised from the bottom to the top of its stroke it will be seen, from the description given above, that as it rises it carries up some of the oil with it as a layer on its upper surface, and after cutting off the communication between the cylinder and the receiver through the tube E, it forces out the air between it and the partition D, through the valve at *a*. The oil on the piston also passes through the valve, and sweeps out every trace of air from the space between the piston and the partition. Then, at the downstroke, a vacuum is formed in the space between the piston and the partition until the tube at E is cleared, and the receiver is put

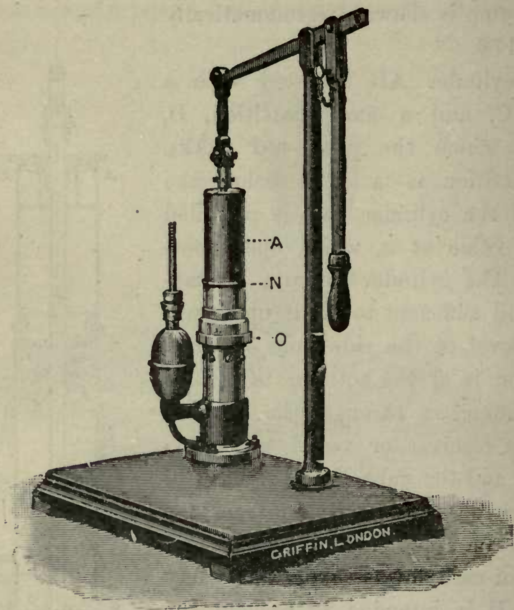


Fig. 177.

in communication with this space. The air in the receiver then expands into the cylinder, and the air thus withdrawn from the receiver is swept out of the cylinder at the next upstroke. During the formation of the vacuum between the piston and partition during the first part of the downstroke, the oil which found its way through the valve *a*, at the end of the upstroke, effectually prevents leakage at this valve, or at the point where the piston-rod passes through the partition.

In this form of pump the motion of the piston is comparatively free from friction, and the pump is very easily worked. When in good working order it readily gives a vacuum in which the pressure is less than that due to a fifth of a millimetre of mercury. A convenient form of the pump is shown in Fig. 177.

A mechanical **compression pump** or syringe, such as may be used for compressing air into a reservoir, is shown diagrammatically in Fig. 178. The action of the pump can readily be

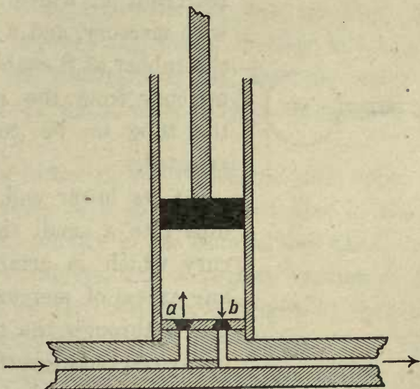


Fig. 178.

followed from the figure; the valve at *a* opens inwards, and the valve at *b* outwards from the cylinder.

129. **Mercury Air Pumps.**—The most satisfactory form of air pump for obtaining a very low vacuum is a mercury pump. In this form of pump the piston is replaced by mercury which is made to rise or fall in the tube or barrel of the instrument, and the use of valves is dispensed with entirely. A mercury pump is called a *lift* pump or a *fall* pump, according as the air is driven out of the body of the pump by the rise or fall of the mercury. In both cases the principle of action is the principle of

Toricelli's experiment, and the vacuum produced by the pump is essentially the Toricellian vacuum.

The typical fall pump is the *Sprengel pump*, first described by Sprengel in 1865. In its original and simplest form this pump is arranged as shown in Fig. 179. The essential part is a long capillary tube, AB, about 900 mm. long, and with a bore of about 1·5 mm. diameter. This tube is connected at A by a stout rubber tube connection with a funnel or reservoir, F, which can be filled with mercury, and a screw clip on the rubber at S enables the fall of mercury from the reservoir into the tube to be controlled and arrested.

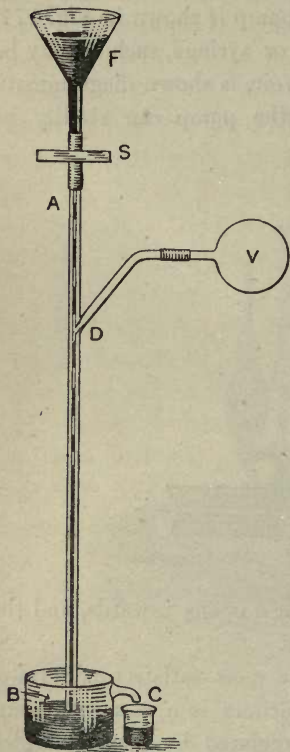


Fig. 179.

At its lower end, B, the tube dips into a small cistern of mercury which is arranged so that any excess of mercury which falls into it through the tube from the upper reservoir overflows into the small vessel C, and can be returned to the reservoir.

At a point, D, about 100 mm. below A, the side tube branches off, as shown in the figure, from the tube AB. This tube communicates with the vessel which is to be exhausted of air by the action of the pump.

When the reservoir F is filled with mercury, and the clip at S opened, the mercury *falls* through the tube AB into the cistern below. The tube being, however, a capillary tube, the mercury does not fall in a continuous stream, but in a sequence of drops which fall through the tube as a sequence of short columns or threads of mercury separated by longer columns of air, as indicated in the figure. Hence, as the mercury continues to fall through the tube, air is continuously removed from the tube and the vessel communicating with it through the side tube. This air is carried down the tube between the successive mercury drops into the cistern, and finds its way thence into the outer air. When the space between any two falling drops comes opposite the opening of the tube at D, the air in the vessel V expands into it, and some of the air from the vessel is carried down between the drops as they fall below D. This process goes on continuously as the successive air spaces pass the point D, until, ultimately, a vacuum is produced in the vessel.

As a vacuum forms in the vessel and the upper part of the tube, mercury rises as a continuous column in the tube above the level of the mercury in the cistern, and when the process of exhaustion is complete, and a perfect vacuum is formed, this column stands at the barometric height in the tube with its upper level a little below the point D. The clip at S is closed,* and at this stage the pump is practically a simple barometer, as in Toricelli's experiment with the Toricellian vacuum in the closed space above the mercury in the tube.

In the simple form described above Sprengel's pump is subject to a serious defect due to the fact that the mercury falling from the reservoir at F carries down air with it and makes it impossible to obtain a good vacuum. This defect is most satisfactorily

* When the vacuum is nearly complete before S is closed, the drops of mercury fall from the reservoir on to the mercury and glass below with a sharp metallic clink due to the absence of the air, which serves, when present, as a cushion or buffer between the drop and the surface on which it falls.

removed by the arrangement shown in Fig. 180, which represents the form in which the pump is now used.

The capillary tube AB is arranged much as in the earlier form of the pump. The reservoir F is not, however, connected directly to the head of the tube, but communicates with it through a long U-tube, G, a bulb, K, which is exhausted of air, and a second long U-tube, H. The side tube DE communicating with the vessel to be exhausted branches off

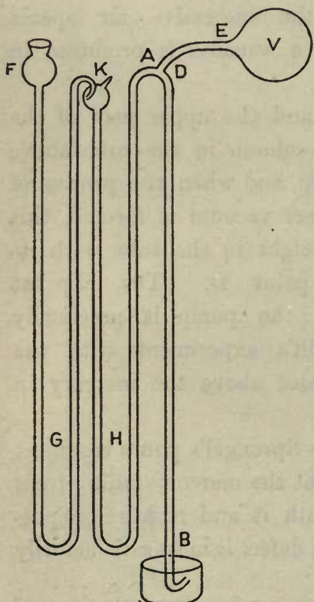


Fig. 180.

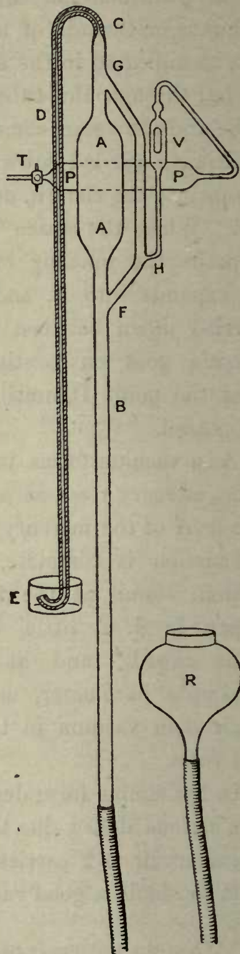


Fig. 181.—Töpler Pump.

in this form of the pump at the head of the tube AB at or near the bend where the tube joins the U-tube H.

The presence of the vacuous bulb at K is found to prevent any air from finding its way with the mercury into the pump. The mercury flows from the tube G into the bulb K as a fine jet which strikes on the side of the bulb; from the bulb K it passes through the U-tube H into the head of the pump at A. Here, as in the capillary tube, it breaks into drops, and carries the air from the vessel V down with it, as explained above. The U-tubes G and H are evidently needed to protect the vacuum at K and the vacuum in V when it is formed. The reader will find it a good exercise to draw diagrams showing the levels of the mercury in the pump at the beginning and at the end of the process of forming a vacuum in V.

One of the most convenient forms of mercury pump is, however, the *lift* pump in the form devised by Töpler, and known generally as **Töpler's pump.**

The general arrangement of this pump and the relative proportion of its parts is shown in Fig. 181. The barrel or body of the pump is formed by the cylindrical reservoir A, which is about 200 mms. long and 50 mms. in diameter. This cylinder is continued downwards into the long tube B, about 800 mms. in length and 13 mms. in diameter. Upwards to the point C, for a short length of about 50 mms., the cylinder is continued by a similar tube, but at this point it joins the long capillary tube D, and the bore narrows gradually and evenly to about 1 mm. This capillary tube is about 800 mms. long, and dips into a small cistern of mercury at E. The side tube FG, also of 13-mm. bore, is arranged in parallel with the cylinder A, and serves to put the cylinder in communication with the vessel to be exhausted through the side tube H, the valve B, and the drying bulb PP. The tube H is of small bore, not more than 4 mms. or 5 mms., and branches off from FG at a point on a level with the lower end of the cylinder. The float valve at V is provided to prevent the mercury as it rises in the cylinder finding its way into the drying bulb; the upper end of the cylindrical float is ground to fit accurately into the tapering bore of the tube at the junction just above it. The drying bulb PP contains phosphorus pentoxide as a drying agent, and communicates through the stopcock at T with the vessel to be exhausted of air.

A large reservoir, R, containing mercury is connected by about a metre length of thick rubber tubing to the lower end of the tube B, and is arranged so that it can be raised or lowered by hand or by means of a suitable lift.

The action of the pump is simpler than the construction of it. The reservoir R is raised so that the mercury rises slowly in the pump until it reaches the point C at the head of the pump. It will be seen that when the mercury in the side tube passes the point H communication with the vessel to be exhausted is cut off, and the further rise of the mercury drives all the air in the cylinder A and in tube FG out through the capillary tube D. The air thus expelled from the pump bubbles out through the mercury in the cistern at E and escapes into the outer air. When the mercury reaches the point C a small quantity is allowed to overflow through the capillary tube, and the reservoir is then carefully lowered. As the mercury falls a vacuum is produced in the upper part of the cylinder and the side tube, but at a certain point air from the vessel being exhausted finds its way through the side tube H and up the tube HG into this vacuum. Hence, by the time the mercury falls below the point F, the air in the vessel will have expanded into the cylinder A and the tubes connected with it. This air may be swept out as before by again raising the reservoir until the mercury overflows at C, and the process of exhaustion may thus be continued by alternately raising and lowering the reservoir until the required degree of exhaustion is obtained.

As the vacuum is formed in the pump and the vessel connected to it, a continuous column of mercury rises in each of the tubes B and D, and when the vacuum is complete the mercury column in each tube stands, subject to the correction for capillarity in each case, at the barometric height.

This form of pump acts quickly and efficiently, but it requires careful handling in raising and lowering the mercury.

130. **Compressed Air Manometer.**—The mercury pressure gauge or manometer has been described in Art. 121. This form of gauge is suitable for pressures less than the atmospheric pressure, or for pressures not greater than two atmospheres; for higher pressures the mercury column in the gauge becomes inconveniently long, and for very high pressures it would be unworkable.

The compressed air manometer is therefore generally used for the measurement of pressures greater than the atmospheric

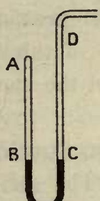


Fig. 182.

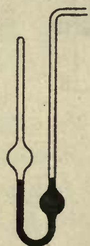


Fig. 183.

pressure, and particularly for high pressures. Its principle of action is derived from Boyle's law; it is, in fact, a form of Boyle's tube. A common form of the instrument is shown in Fig. 182. A column of dry air is confined in the closed limb AB by the mercury in the bend BC, and the open limb CD communicates with the vessel in which pressure is to be measured. The pressure thus applied to the surface of the mercury at C compresses the air in the closed limb AB, and the extent to which the air is thus compressed indicates the pressure to which it is exposed. The tube AB is graduated, in accordance with Boyle's law, to indicate the pressure to which the air in it is exposed, and the reading on the scale of graduation goes directly the pressure in the vessel with which CD is in communication. In some forms of the instrument the bore of the limb BA tapers toward the upper end A; this prevents the rapid decrease in the length of the scale divisions as the pressure increases, and gives, therefore, a more open scale at high pressures.

If the manometer is to be used for the measurement of high pressures only, it is sometimes constructed with a bulb at the lower end of the closed limb AB. As the air in the limb is compressed the air from the bulb is forced into the tube above it, so that when the mercury enters the tube the air above it will be at a pressure which depends in value upon the relative capacities of the bulb and the tube. The tube is then graduated to indicate pressures above this value as the zero of the scale. An example of this form of the manometer is shown in Fig. 183.

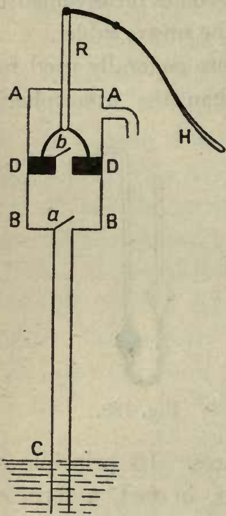


Fig. 184.

131. The Common Suction Pump and Force Pump.—The action of the suction pump commonly used for pumping water up from a well is very similar to that of the air pump shown in Fig. 172. A diagrammatic section of the pump as usually constructed is shown in Fig. 184. The barrel of the pump AB is continued downwards from its lower end by the tube BC which communicates with the water supply, the valve *a* at the junction of the tube with the barrel being made to open upwards. The piston D is constructed to work smoothly and tightly in the barrel and is fitted at *b* with a valve which opens upwards. In the case of a hand pump the piston-rod R is connected to a lever which is provided with a handle at H.

It will be seen that when the piston is set in motion the pump at first acts as an air pump, and by pumping air out of the barrel and the tube BC rapidly decreases the pressure of the air on the surface of the water inside the tube at C. The pressure of the air is therefore greater on the surface of the water outside the tube than on the surface inside the tube, and this excess of pressure forces the water to rise in the tube. As the pressure inside the tube is decreased by the action of the pump the water rises higher and higher until it enters the barrel at the valve *a*. The pump then begins to act as a water pump in much the same way as it at first acted as an air pump. At each down-stroke the water in the barrel is forced above the piston through the valve at *b*; then, at the following up-stroke, this water is lifted by the piston as it rises, and flows out through the spout shown in the figure. A quantity of water determined by the capacity of the barrel is thus delivered from the spout at each up-stroke of the piston.

It will be seen from what has been said that a suction pump cannot work unless the vertical height of the tube BC, from the surface of the water at *c* to the valve at *a*, is less than the height of the water barometer. This height is about 34 feet, so that a suction pump cannot raise water from a greater depth than 34 feet.

The piston of an ordinary water pump does not usually fit well enough for the pump to act effectively as an air pump when there is no water in the barrel; it is therefore sometimes necessary to pour water into the barrel to set the pump in action.

The general action of the **force pump** is very similar to that of the suction pump, but, as shown in Fig. 185, the outlet valve, *b*, from the barrel, is at the mouth of the delivery tube S and not in the piston. The water in the barrel is thus, at each down-stroke, forced through the valve *b* into the delivery tube, and at each up-stroke the back pressure of this water closes the valve and water enters the barrel through the valve at *a*.

Water is thus forced out through the delivery pipe S at each down-stroke of the piston, and the delivery is, therefore, intermittent. If, however, an air chamber A is connected with the delivery pipe as shown diagrammatically in Fig. 186, the air in the chamber becomes compressed under the pressure to which it is exposed, and the steady pressure of the compressed air on the water maintains a steady, continuous delivery from the pump.

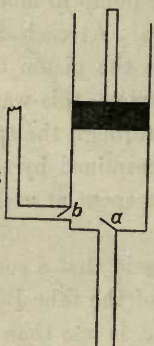


Fig. 185.

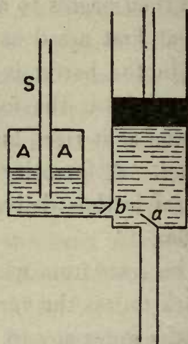


Fig. 186.

132. **The Siphon.**—A siphon is a bent tube used for drawing off liquid from a vessel when the ordinary process of pouring the liquid off is undesirable or cannot be conveniently adopted. The tube usually takes the form of an inverted **U** or **V**, as shown at ABC in Fig. 187, and is generally made with one leg, AC, longer than the other. It is used as a syphon by first filling it with the liquid to be drawn off, and then placing it with the short leg dipping below the level of the liquid in the vessel to be emptied; when this is done the liquid at once flows through the tube and is delivered in a steady stream at the open end of the long leg.

The theory of the action of the instrument is readily understood. Consider the pressure on each face of a very thin transverse slice of the liquid at the highest point, A, of the tube. The pressure on the face in contact with the liquid in the leg AB is evidently the atmospheric pressure minus the pressure due to a column of liquid of height equal to the vertical height of the point A above the level of the liquid at B. Similarly, the pressure on the surface in contact with the liquid in the leg AC is the atmospheric pressure minus the pressure due to a column of liquid of height equal to the vertical height of the point A above the open end of the tube at C. Hence, if h and h' denote the vertical heights of the point A above the points B and C respectively, as indicated in the figure, we have $(P - hdg)$ as the pressure on the face of the slice in contact with the liquid in the leg AB, and $(P - h'dg)$ as the pressure on the face in contact with the liquid in leg AC, where P denotes the atmospheric pressure and d the density of the liquid. If h is less than h', $(P - hdg)$ is obviously greater than $(P - h'dg)$, and the difference of the pressures on the two faces of the slice urging it from the leg BA into the leg AC is given by $(P - hdg) - (P - h'dg)$, or $(h' - h)\,dg$. The difference of pressure which causes the liquid at A to flow outwards through the siphon is thus seen to be proportional to the difference between the level of the liquid at B and the open end of the tube at C.

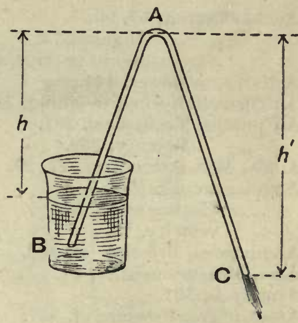


Fig. 187.

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